

Global Well-Posedness with Large Oscillations and Vacuum to the Three-Dimensional Equations of Compressible Nematic Liquid Crystal Flows*

Jing Li

Institute of Applied Mathematics, AMSS & Hua Loo-Keng Key Laboratory of Mathematics,
Chinese Academy of Sciences, Beijing 100190, P. R. China
E-mail: aijingli@gmail.com

Zhonghai Xu

College of Science, Northeast Dianli University, Jilin 132013, P. R. China
E-mail: xuzhonghai@163.com

Jianwen Zhang

School of Mathematical Sciences, Xiamen University, Xiamen 361005, P. R. China
E-mail: jwzhang@xmu.edu.cn

Abstract. This paper is concerned with the three-dimensional equations of a simplified hydrodynamic flow modeling the motion of compressible, nematic liquid crystal materials. The authors establish the global existence of classical solution to the Cauchy problem with smooth initial data which are of small energy but possibly large oscillations with constant state as far-field condition which could be either vacuum or non-vacuum. The initial density is allowed to vanish and the spatial measure of the set of vacuum can be arbitrarily large, in particular, the initial density can even have compact support. As a byproduct, the large-time behavior of the solution is also studied.

Keywords. Compressible nematic liquid crystal flow; Vacuum; Large oscillations; Global well-posedness; Large-time behavior

1 Introduction

Nematic liquid crystals are aggregates of molecules which possess same orientational order and are made of elongated, rod-like molecules. The continuum theory of liquid crystals was established by Ericksen [7] and Leslie [25] during the period of 1958 through 1968, see also the book by de Gennes [11]. Since then, there have been some important research developments in liquid crystals from both theoretical and applied aspects. When the fluid containing nematic liquid crystal materials is at rest, we have the well-known Oseen-Frank theory for static nematic liquid crystals, see the pioneering work by Hardt-Lin- Kinderlehrer [13] on the analysis of energy minimal configurations of nematic liquid crystals. In general, the motion of fluid always takes

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place. The so-called Ericksen-Leslie system is a macroscopic continuum description of the time evolution of the materials under the influence of both the flow velocity field u and the macroscopic description of the microscopic orientation configurations d of rod-like liquid crystals.

When the fluid is an incompressible viscous fluid, Lin [27] first derived a simplified Ericksen-Leslie equations modeling the liquid crystal flows in 1989. Later, Lin and Liu [28, 29] made some important analytic studies, such as the existence of weak/strong solutions and the partial regularity of suitable solutions of the simplified Ericksen-Leslie system, under the assumption that the liquid crystal director field is of either varying length by Leslie's terminology or variable degree of orientation by Ericksen's terminology. Recently, Wang [38] obtained the global well-posedness of the hydrodynamic flow of nematic liquid crystals in the entire space under some small conditions. However, when the fluid is compressible, the Ericksen-Leslie system becomes more complicate and there are very few analytic works available yet. The readers can refer to the recent works due to Morro [35] and Zakharov-Vakulenko [39] on the modeling and numerical studies, respectively.

In this paper, we consider a simplified version of Ericksen-Leslie equations modeling the hydrodynamic flow of compressible, nematic liquid crystals in the whole spatial domain \mathbb{R}^3 :

$$\rho_t + \operatorname{div}(\rho u) = 0, \quad (1.1)$$

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = \mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u - \nabla d \cdot \Delta d, \quad (1.2)$$

$$d_t + u \cdot \nabla d = \Delta d + |\nabla d|^2 d, \quad (1.3)$$

where $\rho : \mathbb{R}^3 \times [0, \infty) \rightarrow \mathbb{R}^+$ is the density of the fluid, $u : \mathbb{R}^3 \times [0, \infty) \rightarrow \mathbb{R}^3$ is the velocity field of the fluid, $P : \mathbb{R}^3 \times [0, \infty) \rightarrow \mathbb{R}^+$ is the pressure of the fluid, and $d : \mathbb{R}^3 \times [0, \infty) \rightarrow \mathbb{S}^2$ (the unit sphere in \mathbb{R}^3) is the unit-vector field that represents the macroscopic/continuum molecular orientations. The viscosity coefficients μ, λ satisfy the physical conditions:

$$\mu > 0, \quad \lambda + \frac{2}{3}\mu \geq 0. \quad (1.4)$$

The pressure $P(\rho)$ is usually determined through the equation of states. Here, we focus our interest on the case of isentropic flows and assume that

$$P(\rho) \triangleq A\rho^\gamma \quad \text{with} \quad A > 0, \gamma > 1. \quad (1.5)$$

Though the system (1.1)–(1.3) is a simplified version, it still retains most of the interesting mathematical properties (without destroying the basic nonlinear structure) of the origin Ericksen-Leslie model for the hydrodynamics of nematic liquid crystals (cf. [6, 8, 12, 25, 27]). From the viewpoint of partial differential equations, (1.1)–(1.3) is a coupled hyperbolic-parabolic system with strong nonlinearities, and thus, its mathematical analysis is full of challenge, especially in the case that vacuum states are allowed (i.e. the density ρ may vanish). It is worth pointing out that when $d \equiv \text{Constant}$, (1.1)–(1.3) reduces to the famous Navier-Stokes equations which describe the three-dimensional motion of compressible viscous isentropic flows and have been studied by many authors, see, for example, [9, 17, 32, 34], and the references cited therein.

We shall look for the solutions, $(\rho(x, t), u(x, t), d(x, t))$, to the Cauchy problem of (1.1)–(1.5) with the far-field behavior

$$(\rho, u, d)(x, t) \rightarrow (\tilde{\rho}, 0, \mathbf{1}) \quad \text{as} \quad |x| \rightarrow \infty, \quad t > 0, \quad (1.6)$$

and the initial data

$$(\rho, u, d)(x, 0) = (\rho_0, u_0, d_0)(x), \quad x \in \mathbb{R}^3, \quad (1.7)$$

where $\tilde{\rho} \geq 0$ is a given nonnegative constant, “ $\mathbf{1}$ ” is a given unit vector and $|d_0| \equiv 1$.

The local-in-time strong solutions to the initial value or initial-boundary value problem of (1.1)–(1.3) with nonnegative initial density were studied by Ding-Lin-Wang-Wen [5] and Huang-Wang [20] in 1-D and 3-D spatial domain, respectively. Recently, based on the arguments in [18, 19] for the compressible Navier-Stokes equations, some interesting results on the blow-up criterion of strong solutions of (1.1)–(1.3) were obtained (see [21, 22, 33]). However, to the author’s knowledge, the global existence of strong/classical solutions with vacuum is still open even that the initial data are suitably small in some sense.

Recently, Huang-Li-Xin [17] established the global existence of classical solution with large oscillations and vacuum to the compressible isentropic Navier-Stokes equations. Motivated by [17], in this paper, we will study the global well-posedness of strong/classical solutions to the Cauchy problem (1.1)–(1.7) when the initial data are sufficiently smooth and are suitably small in some energy-norm.

Before formulating our main results, we first explain the notations and conventions used throughout this paper. For simplicity, we set

$$\int f dx = \int_{\mathbb{R}^3} f dx.$$

For $1 \leq r \leq \infty$ and $\beta > 0$, the standard homogeneous and inhomogeneous Sobolev spaces are simply denoted as follows:

$$\begin{cases} L^r = L^r(\mathbb{R}^3), & D^{k,r} = \{u \in L^2_{\text{loc}} \mid \|\nabla^k u\|_{L^r} < \infty\}, & \|u\|_{D^{k,r}} = \|\nabla^k u\|_{L^r}, \\ W^{k,r} = L^r \cap D^{k,r}, & H^k = W^{k,2}, & D^k = D^{k,2}, & D^1 = \{u \in L^6 \mid \|\nabla u\|_{L^2} < \infty\}, \\ \dot{H}^\beta = \left\{ f : \mathbb{R}^3 \rightarrow \mathbb{R} \mid \|f\|_{\dot{H}^\beta}^2 = \int |\xi|^{2\beta} |\hat{f}(\xi)|^2 d\xi < \infty \right\}, \end{cases}$$

where \hat{f} denotes the Fourier transform of f .

For given initial data (ρ_0, u_0, d_0) , we define

$$C_0 \triangleq \int \left(G(\rho_0) + \frac{1}{2} \rho_0 |u_0|^2 + \frac{1}{2} |\nabla d_0|^2 \right) dx,$$

where $G(\cdot)$ is the potential energy density given by

$$G(\rho) \triangleq \rho \int_{\tilde{\rho}}^{\rho} \frac{P(s) - P(\tilde{\rho})}{s^2} ds.$$

It is clear that

$$\begin{cases} G(\rho) = \frac{1}{\gamma-1} P(\rho), & \text{if } \tilde{\rho} = 0, \\ c_1(\tilde{\rho}, \bar{\rho})(\rho - \tilde{\rho})^2 \leq G(\rho) \leq c_2(\tilde{\rho}, \bar{\rho})(\rho - \tilde{\rho})^2, & \text{if } \tilde{\rho} > 0, 0 \leq \rho \leq 2\bar{\rho}, \end{cases}$$

for positive constants $c_1(\tilde{\rho}, \bar{\rho}), c_2(\tilde{\rho}, \bar{\rho})$ depending on $\tilde{\rho}$ and $\bar{\rho}$.

Our main results in this paper now can be stated as follows.

Theorem 1.1 For given numbers $M_1, M_2 > 0$ (not necessarily small), $\bar{\rho} \geq \tilde{\rho} + 1$, $\beta \in (1/2, 1]$, and $q \in (3, 6)$, assume that the initial data (ρ_0, u_0, d_0) satisfy

$$\begin{cases} G(\rho_0) + \rho_0|u_0|^2 + |\nabla d_0|^2 \in L^1, & 0 \leq \rho_0(x) \leq \bar{\rho}, \\ (\rho_0 - \tilde{\rho}, P(\rho_0) - P(\tilde{\rho})) \in H^2 \cap W^{2,q}, \\ u_0 \in \dot{H}^\beta \cap D^1 \cap D^2, & \nabla d_0 \in \dot{H}^\beta \cap D^1 \cap D^2, \\ \|u_0\|_{\dot{H}^\beta} \leq M_1, & \|\nabla d_0\|_{\dot{H}^\beta} \leq M_2, \quad |d_0| = 1, \end{cases} \quad (1.8)$$

and that the compatibility condition

$$-\mu \Delta u_0 - (\lambda + \mu) \nabla \operatorname{div} u_0 + \nabla P(\rho_0) + \nabla d_0 \cdot \Delta d_0 = \rho_0^{1/2} g \quad (1.9)$$

holds for some $g \in L^2$. Then there exists a positive constant $\varepsilon > 0$, depending only on $\mu, \lambda, A, \gamma, \tilde{\rho}, \tilde{\rho}, M_1$ and M_2 , such that if

$$C_0 \leq \varepsilon, \quad (1.10)$$

the Cauchy problem (1.1)–(1.7) has a unique global classical solution (ρ, u, d) satisfying

$$0 \leq \rho(x, t) \leq 2\bar{\rho} \quad \text{for all } x \in \mathbb{R}^3, t \geq 0, \quad (1.11)$$

and

$$\begin{cases} (\rho - \tilde{\rho}, P(\rho) - P(\tilde{\rho})) \in C([0, T]; H^2 \cap W^{2,q}), \\ u \in C([0, T]; D^1 \cap D^2) \cap L^\infty(\tau, T; D^3 \cap D^{3,q}), \\ u_t \in L^\infty(\tau, T; D^1 \cap D^2) \cap H^1(\tau, T; D^1), \\ \nabla d \in C([0, T]; H^2) \cap L^\infty(\tau, T; H^3), \\ \nabla d_t \in C([0, T]; L^2) \cap H^1(\tau, T; L^2) \end{cases} \quad (1.12)$$

for any $0 < \tau < T < \infty$. Moreover, the following large-time behavior:

$$\lim_{t \rightarrow \infty} \left(\|\rho(\cdot, t) - \tilde{\rho}\|_{L^p} + \int \rho^{1/2} |u|^4 dx + \|\nabla u(\cdot, t)\|_{L^r} + \|\nabla d(\cdot, t)\|_{W^{1,k}} \right) = 0, \quad (1.13)$$

holds for $r \in [2, 6)$, $k \in (2, 6)$, and

$$p \in \begin{cases} (\gamma, \infty) & \text{if } \tilde{\rho} = 0, \\ (2, \infty) & \text{if } \tilde{\rho} > 0. \end{cases} \quad (1.14)$$

Remark 1.1. The solution obtained in Theorem 1.1 becomes a classical one away from the initial time. Moreover, the oscillations of $(\rho, u, \nabla d)$ could be arbitrarily large and the interior and far field vacuum states are allowed.

Remark 1.2. When $d \equiv \text{Constant}$, the system (1.1)–(1.3) reduces to the well-known Navier-Stokes equations for compressible isentropic flows. So, Theorem 1.1 improves the result due to Huang-Li-Xin [17], since the compatibility condition (1.9) imposed on the initial data is much weaker than the one in [17] (see also [4]). Indeed, to prove the existence of a classical solution of the compressible Navier-Stokes equations, the authors in [4, 17] had to assume that

$$-\mu \Delta u_0 - (\mu + \lambda) \nabla \operatorname{div} u_0 + \nabla P(\rho_0) = \rho g \quad \text{with } g \in D^1, \sqrt{\rho_0} g \in L^2. \quad (1.15)$$

Remark 1.3. In [10, 23], Fujita-Kato and Kato proved that the incompressible Navier-Stokes system is globally well-posed for small initial data in the homogeneous Sobolev spaces $\dot{H}^{1/2}$ or in L^3 . In our case, since the initial energy is small, we need the boundedness assumptions on the H^β -norm on the initial velocity which is analogous to the one for the compressible Navier-Stokes equations [17]. Note that $\dot{H}^\beta \hookrightarrow L^{6/(3-2\beta)}$ and $6/(3-2\beta) > 3$ for $\beta > 1/2$. Thus, compared with the results in [10, 23], the conditions on the initial velocity may be optimal under the smallness conditions on the initial energy.

Remark 1.4. Recently, Wang [37, 38] proved that the heat flow of harmonic maps (1.3) (with $u = 0$) and the incompressible liquid crystal flow are globally well-posed provided that $\|\nabla d_0\|_{L^3}$ and $\|u_0\|_{L^3} + \|\nabla d_0\|_{L^3}$ are sufficiently small, respectively. In view of these results in [37, 38], the conditions imposed on the initial director field d_0 may also be optimal under the smallness conditions on the initial energy.

We now comment on the analysis of this paper. For large initial data satisfying (1.8) and (1.9), one can utilize the Galerkin approximation method to construct the local classical solutions in a similar manner as that in [21] (see, Proposition 5.1 below). So, to extend the classical solution globally in time, we need some global a priori estimates on the solutions (ρ, u, d) in suitable higher norms. To do so, we notice that (1.1)–(1.3) are indeed a coupled system between the Navier-Stokes equations and the equations for heat flow of harmonic maps, so that, we shall make use of some ideas developed in [14, 17]. However, compared with the compressible Navier-Stokes equations, some new difficulties arise due to the additional presence of the liquid crystal director field d and the weaker compatibility condition (1.9) (cf. (1.15)). Especially, the super critical nonlinearity $|\nabla d|^2 d$ in the transported heat flow of harmonic map equation (1.3) and the strong coupling nonlinear term $\nabla d \cdot \Delta d$ in the momentum equations (1.2) will cause serious difficulties in the proofs of the time-independent global energy estimates.

As that in [17], it turns out that the key issue in this paper is to prove both the time-independent upper bound for the density and the time-dependent higher norm estimates of the solutions (ρ, u, d) . For this purpose, as usual we start with the basic energy estimate (see Lemma 3.1). To overcome the difficulties induced by the nonlinearities $|\nabla d|^2 d$ and $\nabla d \cdot \Delta d$, we succeed in deriving an estimate on the spatial L^3 -norm of the gradient of d and the estimates on the second-order spatial derivatives of d , which are indeed suitably controlled by the initial energy C_0 and the \dot{H}^β -norm of ∇d (see Lemmas 3.2, 3.3). Then, basing on these estimates, we can utilize the techniques in [14, 15, 17] to obtain an estimate on the spatial L^3 -norm of the velocity, and to carry out some careful initial-layer analysis which is concerned with the elegant estimates on the gradient of the velocity, the second-order and third-order derivatives of the director field, and the material derivatives of the velocity as well (see Lemmas 3.4–3.7). It is worth pointing out that similar to that for the compressible Navier-Stokes equations, the effective viscous flux and the vorticity (see (2.3) for the definition) play a very mathematically important role in the entire analysis, which are instrumental in controlling $\|\nabla u\|_{L^p}$ ($2 \leq p \leq 6$) by the L^2 -norm of the gradient and the material derivative of the velocity (see Lemma 2.2). With these estimates, we then can obtain the desired estimates on both $L^1(0, \min\{1, T\}; L^\infty)$ and $L^{8/3}(\min\{1, T\}, T; L^\infty)$ norms of the effective viscous flux, and thus, it follows from Zlotnik's inequality (see Lemma 2.3) that the density admits a uniform-in-time upper bound which is the cornerstone for the global classical estimates of the solutions.

The next main step is to estimate the derivatives of the solutions. Indeed, to achieve these, we first apply the Beale-Kato-Majda type inequality (see Lemma 2.4) to prove the important estimates on the gradients of the density and velocity by solving a logarithm Gronwall inequality

in a similar manner as that in [17, 19]. As a result, one can easily obtain the L^2 -estimates for the second-order derivatives of density, pressure and velocity. However, due to the weaker compatibility condition in (1.9) (cf. (1.15)), the method used in [17, 19] cannot be applied any more to obtain further estimates. Motivated by [16, 26], instead of the L^2 -method, we succeed in achieving these classical estimates by proving some desired L^q -estimates ($3 < q < 6$) on the higher-order time-spatial derivatives of the density and velocity, basing on some careful initial-layer analysis (see Lemmas 4.4–4.6).

The rest of this paper is organized as follows. In Sect. 2, we first collect some known inequalities and facts which will be frequently used later. In Sect. 3, we derive the time-independent (weighted) energy estimates of the solutions and the key pointwise upper bound of the density, which will be used to study the large-time behavior. In Sect. 4, we establish the time-dependent estimates on the higher-order norms of the solutions, which are needed for the existence of classical solutions. Finally, the proof of the main result (i.e. Theorem 1.1) will be done in Sect. 5

2 Auxiliary lemmas

In this section, we recall some elementary inequalities and known results which will be used frequently later. We start with the well-known Sobolev inequalities (see, for example, [24]).

Lemma 2.1 *For $2 \leq p \leq 6$, $1 < q < \infty$ and $3 < r < \infty$, there exists a generic constant $C > 0$, depending only on q and r , such that for $f \in H^1$ and $g \in L^q \cap D^{1,r}$, we have*

$$\|f\|_{L^p} \leq C \|f\|_{L^2}^{(6-p)/(2p)} \|\nabla f\|_{L^2}^{(3p-6)/(2p)}, \quad (2.1)$$

$$\|g\|_{L^\infty} \leq C \|g\|_{L^q}^{q(r-3)/(3r+q(r-3))} \|\nabla g\|_{L^r}^{3r/(3r+q(r-3))}. \quad (2.2)$$

As that for compressible Navier-Stokes equations (see, for example, [14, 17, 32]), the connections among the effective viscous flux, the vorticity and the physical quantities will play an important role in the entire analysis of the present paper. So, to be continued, we set

$$F \triangleq (2\mu + \lambda)\operatorname{div} u - (P(\rho) - P(\tilde{\rho})), \quad \omega \triangleq \nabla \times u, \quad M(d) \triangleq \nabla d \odot \nabla d - \frac{1}{2}|\nabla d|^2 \mathbb{I}_3, \quad (2.3)$$

where F is the so-called effective viscous flux, ω is the vorticity, \mathbb{I}_3 is the 3×3 unit matrix, and

$$\nabla d \odot \nabla d = \sum_{k=1}^3 \partial_i d^k \partial_j d^k.$$

Then, it follows from Lemma 2.1 and the standard L^p -estimates of elliptic system that

Lemma 2.2 *Let (ρ, u, d) be a smooth solution of (1.1)–(1.7) on $\mathbb{R}^3 \times (0, T]$. Then there exists a generic constant $C > 0$, which may depend on μ and λ , such that for any $p \in [2, 6]$,*

$$\|\nabla F\|_{L^p} + \|\nabla \omega\|_{L^p} \leq C (\|\rho \dot{u}\|_{L^p} + \|\nabla d\|_{L^p} \|\nabla^2 d\|_{L^p}), \quad (2.4)$$

$$\|F\|_{L^6} + \|\omega\|_{L^6} \leq C (\|\rho \dot{u}\|_{L^2} + \|\nabla d\|_{L^2} \|\nabla^2 d\|_{L^2}), \quad (2.5)$$

and

$$\|\nabla u\|_{L^6} \leq C (\|\rho \dot{u}\|_{L^2} + \|P(\rho) - P(\tilde{\rho})\|_{L^6} + \|\nabla d\|_{L^2} \|\nabla^2 d\|_{L^2}). \quad (2.6)$$

Proof. Indeed, due to (1.2), one has

$$\rho \dot{u} = \nabla F - \mu \nabla \times \omega - \operatorname{div}(M(d)),$$

and hence,

$$\Delta F = \operatorname{div}(\rho \dot{u}) + \operatorname{div} \operatorname{div}(M(d)), \quad \mu \Delta \omega = \nabla \times (\rho \dot{u} + \operatorname{div}(M(d))), \quad (2.7)$$

where $\dot{f} \triangleq f_t + u \cdot \nabla f$ denotes the material derivative. Thus, an application of the standard L^p -estimate of elliptic system leads to (2.4), which, together with (2.1), gives (2.5). Using (2.3), (2.5) and the standard L^p -estimate, we obtain (2.6). \square

To prove the uniform-in-time upper bound of density, we need the following Zlotnik inequality, whose proof can be found in [40].

Lemma 2.3 *Assume that the function $y \in W^{1,1}(0, T)$ solves the ODE system:*

$$y' = g(y) + b'(t) \quad \text{on} \quad [0, T], \quad y(0) = y_0,$$

where $b \in W^{1,1}(0, T)$ and $g \in C(\mathbb{R})$. If $g(\infty) = -\infty$ and

$$b(t_2) - b(t_1) \leq N_0 + N_1(t_2 - t_1) \quad (2.8)$$

for all $0 \leq t_1 \leq t_2 \leq T$ with some $N_0 \geq 0$ and $N_1 \geq 0$, then one has

$$y(t) \leq \max\{y_0, \xi^*\} + N_0 < +\infty \quad \text{on} \quad [0, T], \quad (2.9)$$

where $\xi^* \in \mathbb{R}$ is a constant such that

$$g(\xi) \leq -N_1 \quad \text{for} \quad \xi \geq \xi^*. \quad (2.10)$$

Finally, we recall the following Beale-Kato-Majda type inequality (cf. [1, 19]), which is an essential tool for the estimates of the gradients of (ρ, u) .

Lemma 2.4 *For $q \in (3, \infty)$, there exists a constant $C(q) > 0$ such that for all $\nabla u \in L^2 \cap D^{1,q}$,*

$$\|\nabla u\|_{L^\infty} \leq C (\|\operatorname{div} u\|_{L^\infty} + \|\nabla \times u\|_{L^\infty}) \ln(e + \|\nabla^2 u\|_{L^q}) + C \|\nabla u\|_{L^2} + C. \quad (2.11)$$

3 Time-independent lower-order estimates

This section is devoted to proving the time-independent (weighted) energy estimates and the uniform upper bound of density. Assume that (ρ, u, d) is a smooth solution of (1.1)–(1.7) on $\mathbb{R}^3 \times (0, T)$ with some positive time T . To estimate the solutions, we set

$$\sigma(t) \triangleq \{1, t\}$$

and define the following functionals:

$$\begin{aligned} A_1(T) &\triangleq \sup_{0 \leq t \leq T} \sigma \left(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 \right) \\ &\quad + \int_0^T \sigma \left(\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 \right) dt, \end{aligned} \quad (3.1)$$

$$A_2(T) \triangleq \sup_{0 \leq t \leq T} \sigma^2 \left(\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 \right) + \int_0^T \sigma^2 \left(\|\nabla \dot{u}\|_{L^2}^2 + \|d_{tt}\|_{L^2}^2 + \|\nabla^2 d_t\|_{L^2}^2 \right) dt, \quad (3.2)$$

$$A_3(T) \triangleq \sup_{0 \leq t \leq T} \|\nabla d\|_{L^3}^3 + \int_0^T \int |\nabla d| |\nabla^2 d|^2 dx dt \quad (3.3)$$

$$A_4(T) \triangleq \sup_{0 \leq t \leq T} \sigma^{(3-2\beta)/4} \left(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 \right) + \int_0^T \sigma^{(3-2\beta)/4} \left(\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 \right) dt \quad (3.4)$$

and

$$A_5(T) \triangleq \sup_{0 \leq t \leq T} \int \rho |u|^3 dx, \quad (3.5)$$

where $\beta \in (1/2, 1]$ and $\dot{v} = v_t + u \cdot \nabla v$ is the material derivative.

We shall prove the following key a priori estimates on the solutions (ρ, u, d) .

Proposition 3.1 *For given numbers $M_1, M_2 > 0$, $\bar{\rho} \geq \tilde{\rho} + 1$ and $\beta \in (1/2, 1]$, assume that*

$$\begin{cases} G(\rho_0) + \rho_0 |u_0|^2 + |\nabla d_0|^2 \in L^1, \\ 0 \leq \inf \rho_0 \leq \sup \rho_0 \leq \bar{\rho}, \quad |d_0| = 1, \\ \|u_0\|_{\dot{H}^\beta} \leq M_1, \quad \|\nabla d_0\|_{\dot{H}^\beta} \leq M_2. \end{cases} \quad (3.6)$$

Suppose that (ρ, u, d) is a smooth solution of (1.1)–(1.7) on $\mathbb{R}^3 \times (0, T]$ satisfying

$$\begin{cases} 0 \leq \inf_{(x,t) \in \mathbb{R}^3 \times [0,T]} \rho(x,t) \leq \sup_{(x,t) \in \mathbb{R}^3 \times [0,T]} \rho(x,t) \leq 2\bar{\rho}, \\ A_1(T) + A_2(T) \leq 2C_0^{1/2}, \quad A_3(T) \leq 2C_0^{\delta_0}, \quad A_4(\sigma(T)) + A_5(\sigma(T)) \leq 2C_0^{\delta_0}, \end{cases} \quad (3.7)$$

where

$$\delta_0 \triangleq \frac{2\beta - 1}{9\beta} \in (0, 1/9] \quad \text{for } \beta \in (1/2, 1]. \quad (3.8)$$

Then there exists a positive constant $\varepsilon > 0$, which depends only on $\mu, \lambda, A, \gamma, \bar{\rho}, \tilde{\rho}, \beta, M_1$, and M_2 , such that

$$\begin{cases} 0 \leq \inf_{(x,t) \in \mathbb{R}^3 \times [0,T]} \rho(x,t) \leq \sup_{(x,t) \in \mathbb{R}^3 \times [0,T]} \rho(x,t) \leq \frac{7}{4}\bar{\rho}, \\ A_1(T) + A_2(T) \leq C_0^{1/2}, \quad A_3(T) \leq C_0^{\delta_0}, \quad A_4(\sigma(T)) + A_5(\sigma(T)) \leq C_0^{\delta_0}, \end{cases} \quad (3.9)$$

provided

$$C_0 \leq \varepsilon. \quad (3.10)$$

The proof of Proposition 3.1 is based on a series of lemmas and is postponed to the end of this section. For notational convenience, throughout this section we denote by C or C_i ($i = 1, 2, \dots$) the generic positive constants which may depend on $\mu, \lambda, A, \gamma, \bar{\rho}, \tilde{\rho}, \beta, M_1$, and M_2 , but are independent of T . We also sometimes write $C(\alpha)$ to emphasize the dependence on α .

We begin with the following estimates.

Lemma 3.1 *Let (ρ, u, d) be a smooth solution of (1.1)–(1.7) on $\mathbb{R}^3 \times (0, T]$ satisfying (3.7). Then there exists a constant $\varepsilon_1 > 0$, depending only on $\mu, \lambda, A, \gamma, \bar{\rho}, \tilde{\rho}, M_1$, and M_2 , such that*

$$\begin{aligned} & \sup_{0 \leq t \leq T} \int (\rho|u|^2 + G(\rho) + |\nabla d|^2) dx \\ & + \int_0^T (\|\nabla u\|_{L^2}^2 + \|d_t\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2) dt \leq CC_0, \end{aligned} \quad (3.11)$$

and moreover,

$$\int_0^T (\|u\|_{L^6}^4 + \|\nabla u\|_{L^2}^4 + \|\nabla^2 d\|_{L^2}^4) dt \leq CC_0^{2\delta_0}, \quad (3.12)$$

provided $C_0 \leq \varepsilon_1$.

Proof. Multiplying (1.1) and (1.2) by $G'(\rho)$ and u in L^2 respectively, integrating by parts and adding them together, by (2.1) we have

$$\begin{aligned} & \frac{d}{dt} \int \left(G(\rho) + \frac{1}{2} \rho |u|^2 \right) dx + \int (\mu |\nabla u|^2 + (\mu + \lambda) (\operatorname{div} u)^2) dx \\ & = - \int (\nabla d \cdot \Delta d) \cdot u dx \leq C \|u\|_{L^6} \|\nabla d\|_{L^3} \|\nabla^2 d\|_{L^2} \\ & \leq \frac{\mu}{4} \|\nabla u\|_{L^2}^2 + C \|\nabla d\|_{L^3}^2 \|\nabla^2 d\|_{L^2}^2. \end{aligned} \quad (3.13)$$

Using the fact that $|d| = 1$ and integrating by parts, we infer from (1.3) and (2.1) that

$$\begin{aligned} & \frac{d}{dt} \int |\nabla d|^2 dx + \int (|d_t|^2 + |\nabla^2 d|^2) dx \\ & = \int |d_t - \Delta d|^2 dx = \int |u \cdot \nabla d - |\nabla d|^2 d|^2 dx \\ & \leq C (\|u\|_{L^6}^2 \|\nabla d\|_{L^3}^2 + \|\nabla d\|_{L^4}^4) \\ & \leq C \|\nabla d\|_{L^3}^2 (\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2), \end{aligned}$$

which, together with (3.13) and (3.7), gives

$$\begin{aligned} & \frac{d}{dt} \int \left(G(\rho) + \frac{1}{2} \rho |u|^2 + |\nabla d|^2 \right) dx + \left(\frac{3\mu}{4} \|\nabla u\|_{L^2}^2 + \|d_t\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 \right) \\ & \leq C \|\nabla d\|_{L^3}^2 (\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2) \leq C_1 C_0^{2\delta_0/3} (\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2). \end{aligned} \quad (3.14)$$

Thus if C_0 is chosen to be such that

$$C_0 \leq \varepsilon_1 \triangleq \min \left\{ 1, \left(\frac{\mu}{4C_1} \right)^{3/(2\delta_0)}, \left(\frac{1}{2C_1} \right)^{3/(2\delta_0)} \right\},$$

then integrating (3.14) in t over $[0, T]$ immediately leads to (3.11).

To prove (3.12), we utilize (2.1), (3.7) and (3.11) to deduce that

$$\int_0^T (\|u\|_{L^6}^4 + \|\nabla u\|_{L^2}^4 + \|\nabla^2 d\|_{L^2}^4) dt$$

$$\begin{aligned}
&\leq C \int_0^{\sigma(T)} (\|\nabla u\|_{L^2}^4 + \|\nabla^2 d\|_{L^2}^4) dt + \int_{\sigma(T)}^T \sigma (\|\nabla u\|_{L^2}^4 + \|\nabla^2 d\|_{L^2}^4) dt \\
&\leq C \sup_{0 \leq t \leq \sigma(T)} \left(\sigma^{(3-2\beta)/4} (\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2) \right)^2 \int_0^{\sigma(T)} \sigma^{(2\beta-3)/2} dt \\
&\quad + C \sup_{\sigma(T) \leq t \leq T} \sigma (\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2) \int_{\sigma(T)}^T (\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2) dt \\
&\leq CC_0^{2\delta_0} + CC_0^{3/2} \leq CC_0^{2\delta_0},
\end{aligned}$$

since $\beta \in (1/2, 1]$ and $\delta_0 \in (0, 1/9]$. The proof of Lemma 3.1 is therefore completed. \square

The next lemma is concerned with the estimate of $A_3(T)$.

Lemma 3.2 *Let (ρ, u, d) be a smooth solution of (1.1)–(1.7) on $\mathbb{R}^3 \times (0, T]$ satisfying (3.7). Then there exists a constant $\varepsilon_2 > 0$, depending only on $\mu, \lambda, A, \gamma, \bar{\rho}, \tilde{\rho}, \beta, M_1$ and M_2 , such that*

$$A_3(T) + \int_0^T \|\nabla d\|_{L^9}^3 dt \leq C_0^{\delta_0}, \quad (3.15)$$

provided $C_0 \leq \varepsilon_2$.

Proof. Operating ∇ to both sides of (1.3) gives

$$\nabla d_t - \nabla \Delta d = \nabla(|\nabla d|^2 d) - \nabla(u \cdot \nabla d). \quad (3.16)$$

Thus, multiplying (3.16) by $3|\nabla d|\nabla d$ and integrating by parts over \mathbb{R}^3 , we obtain

$$\begin{aligned}
&\frac{d}{dt} \|\nabla d\|_{L^3}^3 + 3 \int \left(|\nabla d| |\nabla^2 d|^2 + |\nabla d| |\nabla(|\nabla d|)|^2 \right) dx \\
&\leq C \int (|\nabla d|^3 |\nabla^2 d| + |u| |\nabla d|^2 |\nabla^2 d|) dx \\
&\leq \frac{1}{2} \int |\nabla d| |\nabla^2 d|^2 dx + C (\|\nabla d\|_{L^5}^5 + \|\nabla u\|_{L^2}^2 \|\nabla d\|_{L^{9/2}}^3). \quad (3.17)
\end{aligned}$$

To deal with the right-hand side of (3.17), we first use Lemma 2.1 to get that

$$\begin{cases} \|\nabla d\|_{L^5} \leq C \|\nabla d\|_{L^3}^{2/5} \|\nabla d\|_{L^9}^{3/5}, & \|\nabla d\|_{L^{9/2}} \leq C \|\nabla d\|_{L^3}^{1/2} \|\nabla d\|_{L^9}^{1/2}, \\ \|\nabla d\|_{L^9}^3 = \| |\nabla d|^{3/2} \|_{L^6}^2 \leq C \|\nabla(|\nabla d|^{3/2})\|_{L^2}^2 \leq C \| |\nabla d|^{1/2} \nabla^2 d \|_{L^2}^2, \end{cases} \quad (3.18)$$

so that, by virtue of (3.7) and Cauchy-Schwarz inequality we find

$$\begin{aligned}
\text{R.H.S. of (3.17)} &\leq (1 + C \|\nabla d\|_{L^3}^2) \| |\nabla d|^{1/2} \nabla^2 d \|_{L^2}^2 + C \|\nabla d\|_{L^3}^3 \|\nabla u\|_{L^2}^4 \\
&\leq \left(1 + C_1 C_0^{2\delta_0/3} \right) \| |\nabla d|^{1/2} \nabla^2 d \|_{L^2}^2 + C C_0^{\delta_0} \|\nabla u\|_{L^2}^4. \quad (3.19)
\end{aligned}$$

Thus if C_0 is chosen to be such that

$$C_0 \leq \varepsilon_{2,1} \triangleq \min \left\{ \varepsilon_1, C_1^{-3/(2\delta_0)} \right\},$$

then we obtain after putting (3.19) into (3.17) and integrating it over $(0, T)$ that

$$\sup_{0 \leq t \leq T} \|\nabla d\|_{L^3}^3 + \int_0^T \left(\|\nabla d\|_{L^9}^3 + \| |\nabla d|^{1/2} \nabla^2 d \|_{L^2}^2 \right) dt$$

$$\begin{aligned}
&\leq C\|\nabla d_0\|_{L^3}^3 + CC_0^{\delta_0} \int_0^T \|\nabla u\|_{L^2}^4 dt \\
&\leq C\|\nabla d_0\|_{L^2}^{3(2\beta-1)/(2\beta)} \|\nabla d_0\|_{\dot{H}^\beta}^{3/(2\beta)} + CC_0^{3\delta_0} \\
&\leq CC_0^{3(2\beta-1)/(4\beta)} + CC_0^{3\delta_0} \leq C_2 C_0^{3\delta_0},
\end{aligned} \tag{3.20}$$

where we have used (3.12), (3.18), (2.1), and the Sobolev embedding inequality $\dot{H}^\beta \hookrightarrow L^{6/(3-2\beta)}$.

Therefore, by choosing C_0 sufficiently small to be such that

$$C_0 \leq \varepsilon_2 \triangleq \min \left\{ \varepsilon_{2,1}, C_2^{-2\delta_0} \right\},$$

we immediately obtain the desired estimate in (3.15) from (3.20). \square

The following initial-layer estimate of d is crucial for the subsequent analysis.

Lemma 3.3 *Let (ρ, u, d) be a smooth solution of (1.1)–(1.7) on $\mathbb{R}^3 \times (0, T]$ satisfying (3.7). Then there exists a positive constant ε_3 , depending only on $\mu, \lambda, A, \gamma, \bar{\rho}, \tilde{\rho}, \beta, M_1$ and M_2 , such that for any $\theta \in [0, 1]$,*

$$\sup_{0 \leq t \leq T} \left(\sigma^{1-\theta} \|\nabla^2 d\|_{L^2}^2 \right) + \int_0^T \sigma^{1-\theta} (\|\nabla d_t\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2) dt \leq C \|\nabla d_0\|_{\dot{H}^\theta}^2, \tag{3.21}$$

provided $C_0 \leq \varepsilon_3$. In particular, if $C_0 \leq \varepsilon_3$, then

$$\sup_{0 \leq t \leq T} (\sigma \|\nabla^2 d\|_{L^2}^2) + \int_0^T \sigma (\|\nabla d_t\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2) dt \leq C \|\nabla d_0\|_{L^2}^2 \leq CC_0. \tag{3.22}$$

Proof. Let (ρ, u, d) be the smooth solution of (1.1)–(1.7) on $\mathbb{R}^3 \times (0, T]$. Consider the following Cauchy problem of linear parabolic equations:

$$v_t - \Delta v = -u \cdot \nabla v + d \nabla d : \nabla v, \quad v(x, 0) = v_0(x), \tag{3.23}$$

where $A : B \triangleq \sum_{i,j=1}^3 a_{i,j} b_{i,j}$ for $A = (a_{i,j})_{3 \times 3}$ and $B = (b_{i,j})_{3 \times 3}$.

By (2.1) and Hölder inequality, we easily deduce from (3.23) that

$$\begin{aligned}
&\frac{d}{dt} \|\nabla v\|_{L^2}^2 + (\|v_t\|_{L^2}^2 + \|\nabla^2 v\|_{L^2}^2) \\
&= \int |v_t - \Delta v|^2 dx = \int |u \cdot \nabla v - d \nabla d : \nabla v|^2 dx \\
&\leq C \|\nabla u\|_{L^2}^2 \|\nabla v\|_{L^2} \|\nabla v\|_{L^6} + C \|\nabla d\|_{L^9}^2 \|\nabla v\|_{L^2}^{4/3} \|\nabla v\|_{L^6}^{2/3} \\
&\leq \frac{1}{2} \|\nabla^2 v\|_{L^2}^2 + C (\|\nabla u\|_{L^2}^4 + \|\nabla d\|_{L^9}^3) \|\nabla v\|_{L^2}^2,
\end{aligned}$$

which, together with Gronwall's inequality, (3.12) and (3.15), gives

$$\sup_{0 \leq t \leq T} \|\nabla v\|_{L^2}^2 + \int_0^T (\|v_t\|_{L^2}^2 + \|\nabla^2 v\|_{L^2}^2) dt \leq C \|\nabla v_0\|_{L^2}^2. \tag{3.24}$$

Applying ∇ to both sides of (3.23) and taking the L^2 -inner product, we get

$$\frac{d}{dt} \|\nabla^2 v\|_{L^2}^2 + (\|\nabla v_t\|_{L^2}^2 + \|\nabla^3 v\|_{L^2}^2)$$

$$\leq \int (|\nabla(d\nabla d : \nabla v)|^2 + |\nabla(u \cdot \nabla v)|^2) dx \triangleq I. \quad (3.25)$$

Using Lemma 2.1, (3.15) and Cauchy-Schwarz inequality, we know that

$$\begin{aligned} I &\leq C \int [(|\nabla d|^2 + |u|^2) |\nabla^2 v|^2 + (|\nabla^2 d|^2 + |\nabla d|^4 + |\nabla u|^2) |\nabla v|^2] dx \\ &\leq C \|\nabla d\|_{L^3}^2 \|\nabla^2 v\|_{L^6}^2 + C \|u\|_{L^6}^2 \|\nabla^2 v\|_{L^2} \|\nabla^2 v\|_{L^6} \\ &\quad + C \|\nabla v\|_{L^\infty}^2 (\|\nabla^2 d\|_{L^2}^2 + \|\nabla d\|_{L^3}^2 \|\nabla d\|_{L^6}^2 + \|\nabla u\|_{L^2}^2) \\ &\leq \left(\frac{1}{4} + C_2 C_0^{2\delta_0/3} \right) \|\nabla^3 v\|_{L^2}^2 + C \|\nabla^2 v\|_{L^2}^2 (\|\nabla^2 d\|_{L^2}^4 + \|\nabla u\|_{L^2}^4). \end{aligned} \quad (3.26)$$

Thus if C_0 is chosen to be such that

$$C_0 \leq \varepsilon_3 \triangleq (4C_2)^{-3/(2\delta_0)},$$

then substitution of (3.26) into (3.25) results in

$$\frac{d}{dt} \|\nabla^2 v\|_{L^2}^2 + (\|\nabla v_t\|_{L^2}^2 + \|\nabla^3 v\|_{L^2}^2) \leq C \|\nabla^2 v\|_{L^2}^2 (\|\nabla^2 d\|_{L^2}^4 + \|\nabla u\|_{L^2}^4), \quad (3.27)$$

which, together with Gronwall's inequality and (3.12), implies that

$$\sup_{0 \leq t \leq T} \|\nabla^2 v\|_{L^2}^2 + \int_0^T (\|\nabla v_t\|_{L^2}^2 + \|\nabla^3 v\|_{L^2}^2) dt \leq C \|\nabla^2 v_0\|_{L^2}^2. \quad (3.28)$$

On the other hand, multiplying (3.27) by σ and integrating it over $(0, T)$, we infer from (3.24) that if $C_0 \leq \varepsilon_3$, then

$$\begin{aligned} &\sup_{0 \leq t \leq T} (\sigma \|\nabla^2 v\|_{L^2}^2) + \int_0^T \sigma (\|\nabla v_t\|_{L^2}^2 + \|\nabla^3 v\|_{L^2}^2) dt \\ &\leq C \int_0^T \|\nabla^2 v\|_{L^2}^2 dt \leq C \|\nabla v_0\|_{L^2}^2. \end{aligned} \quad (3.29)$$

Since the solution operator $v_0 \mapsto v(\cdot, t)$ is linear, one may apply the standard Riesz-Thorin interpolation argument (see [2]) to (3.28) and (3.29) to get that for any $\theta \in [0, 1]$,

$$\sup_{0 \leq t \leq T} \left(t^{1-\theta} \|\nabla^2 v\|_{L^2}^2 \right) + \int_0^T t^{1-\theta} (\|\nabla v_t\|_{L^2}^2 + \|\nabla^3 v\|_{L^2}^2) dt \leq C \|\nabla v_0\|_{\dot{H}^\theta}^2, \quad (3.30)$$

with a uniform constant C independent of θ . Thus, choosing $v_0 = d_0$ so that $v = d$, then (3.21) follows from (3.30) directly. \square

By Lemmas 3.1–3.3, we can now derive preliminary bounds for $A_1(T)$ and $A_2(T)$.

Lemma 3.4 *Let (ρ, u, d) be a smooth solution of (1.1)–(1.7) on $\mathbb{R}^3 \times (0, T]$ satisfying (3.7). Then there exists a constant $\varepsilon_4 > 0$, depending only on $\mu, \lambda, A, \gamma, \bar{\rho}, \tilde{\rho}, \beta, M_1$ and M_2 , such that*

$$A_1(T) \leq CC_0 + C \int_0^T \sigma^2 (\|\nabla u\|_{L^4}^4 + \|P(\rho) - P(\tilde{\rho})\|_{L^4}^4) dt, \quad (3.31)$$

and

$$A_2(T) \leq CC_0^{1/2+2\delta_0} + CA_1(T) + C \int_0^T \sigma^2 \|\nabla u\|_{L^4}^4 dt. \quad (3.32)$$

provided $C_0 \leq \varepsilon_4$.

Proof. Multiplying (1.2) by σu_t and integrating by parts over \mathbb{R}^3 give

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int \sigma (\mu |\nabla u|^2 + (\mu + \lambda)(\operatorname{div} u)^2) dx + \sigma \int \rho |\dot{u}|^2 dx \\
&= \frac{d}{dt} \int \sigma ((P(\rho) - P(\tilde{\rho})) \operatorname{div} u + M(d) : \nabla u) dx \\
&+ \sigma \int \rho u \cdot \nabla u \cdot \dot{u} dx - \sigma \int (P(\rho) - P(\tilde{\rho}))_t \operatorname{div} u dx - \sigma \int M(d)_t : \nabla u dx \\
&+ \sigma' \int \left(\frac{1}{2} (\mu |\nabla u|^2 + (\mu + \lambda)(\operatorname{div} u)^2) - (P(\rho) - P(\tilde{\rho})) \operatorname{div} u - M(d) : \nabla u \right) dx \\
&\triangleq \frac{d}{dt} I_0 + \sum_{i=1}^4 I_i.
\end{aligned} \tag{3.33}$$

We are now in a position of estimating the terms on the right-hand side of (3.33). First, by Hölder and Cauchy-Schwarz inequalities we easily see that

$$\begin{aligned}
I_1 &\leq C \sigma \|\rho^{1/2} u\|_{L^4} \|\nabla u\|_{L^4} \|\rho^{1/2} \dot{u}\|_{L^2} \\
&\leq \frac{1}{8} \sigma \|\rho^{1/2} \dot{u}\|_{L^2}^2 + C \sigma^2 \|\nabla u\|_{L^4}^4 + C \|\rho^{1/2} u\|_{L^4}^4.
\end{aligned} \tag{3.34}$$

To deal with I_2 , we first deduce from (1.1) that

$$(P(\rho) - P(\tilde{\rho}))_t + u \cdot \nabla (P(\rho) - P(\tilde{\rho})) + \gamma P(\rho) \operatorname{div} u = 0, \tag{3.35}$$

which, together with the effective viscous flux F in (2.3), yields that

$$\begin{aligned}
I_2 &= \sigma \int \gamma P(\rho) (\operatorname{div} u)^2 dx + \sigma \int u \cdot \nabla (P(\rho) - P(\tilde{\rho})) \operatorname{div} u dx \\
&\leq C \|\nabla u\|_{L^2}^2 + \frac{\sigma}{2\mu + \lambda} \int u \cdot \nabla (P(\rho) - P(\tilde{\rho})) (F + (P(\rho) - P(\tilde{\rho}))) dx \\
&\leq C \|\nabla u\|_{L^2}^2 + C \sigma^2 \|P(\rho) - P(\tilde{\rho})\|_{L^4}^4 + C \sigma \int |P(\rho) - P(\tilde{\rho})| (|\nabla u| |F| + |u| |\nabla F|) dx \\
&\leq C \|\nabla u\|_{L^2}^2 + C \sigma^2 \|P(\rho) - P(\tilde{\rho})\|_{L^4}^4 + C \sigma \|P(\rho) - P(\tilde{\rho})\|_{L^3} \|\nabla u\|_{L^2} \|\nabla F\|_{L^2} \\
&\leq C \|\nabla u\|_{L^2}^2 + C \sigma^2 \|P(\rho) - P(\tilde{\rho})\|_{L^4}^4 + C \sigma \|\nabla u\|_{L^2} \left(\|\rho^{1/2} \dot{u}\|_{L^2} + \|\nabla d\| \|\nabla^2 d\|_{L^2} \right) \\
&\leq \frac{1}{8} \sigma \|\rho^{1/2} \dot{u}\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 + C \sigma \|\nabla^3 d\|_{L^2}^2 + C \sigma^2 \|P(\rho) - P(\tilde{\rho})\|_{L^4}^4,
\end{aligned} \tag{3.36}$$

where we have also used (2.1), (3.11), (3.15) and Cauchy-Schwarz inequality.

It follows from (2.1), (3.15) and Cauchy-Schwarz inequality that

$$\begin{aligned}
I_3 &\leq C \sigma \|\nabla d\|_{L^3}^{1/2} \|\nabla d\|_{L^6}^{1/2} \|\nabla d_t\|_{L^2} \|\nabla u\|_{L^4} \\
&\leq C \|\nabla^2 d\|_{L^2}^2 + C \sigma \|\nabla d_t\|_{L^2}^2 + C \sigma^2 \|\nabla u\|_{L^4}^4,
\end{aligned} \tag{3.37}$$

and finally, it is easily seen that

$$\begin{aligned}
I_4 &\leq C \sigma' (\|P(\rho) - P(\tilde{\rho})\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla d\|_{L^4}^4) \\
&\leq C \sigma' \left(C_0 + \|\nabla u\|_{L^2}^2 + C_0^{2\delta_0/3} \|\nabla^2 d\|_{L^2}^2 \right).
\end{aligned} \tag{3.38}$$

Due to (3.11), (3.15) and (3.22), we have

$$\begin{aligned} I_0 &\leq \frac{\mu}{4} \sigma \|\nabla u\|_{L^2}^2 + C \sigma (\|P(\rho) - P(\tilde{\rho})\|_{L^2}^2 + \|\nabla d\|_{L^3}^2 \|\nabla^2 d\|_{L^2}^2) \\ &\leq \frac{\mu}{4} \sigma \|\nabla u\|_{L^2}^2 + C C_0, \end{aligned} \quad (3.39)$$

so that, putting (3.34) and (3.36)–(3.39) into (3.33) and integrating it over $(0, T)$, we obtain

$$\begin{aligned} &\sup_{0 \leq t \leq T} (\sigma \|\nabla u\|_{L^2}^2) + \int_0^T \sigma \|\rho^{1/2} \dot{u}\|_{L^2}^2 dt \\ &\leq C C_0 + C \int_0^T (\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 + \sigma \|\nabla^3 d\|_{L^2}^2 + \sigma \|\nabla d_t\|_{L^2}^2) dt \\ &\quad + C \int_0^T \sigma^2 (\|\nabla u\|_{L^4}^4 + \|P(\rho) - P(\tilde{\rho})\|_{L^4}^4) dt + C \int_0^T \|\rho^{1/2} u\|_{L^4}^4 dt \\ &\leq C C_0 + C \int_0^T \sigma^2 (\|\nabla u\|_{L^4}^4 + \|P(\rho) - P(\tilde{\rho})\|_{L^4}^4) dt + C \int_0^T \|\rho^{1/2} u\|_{L^4}^4 dt, \end{aligned} \quad (3.40)$$

provided $C_0 \leq \varepsilon_3$. Here we have also used (3.11) and (3.22).

To estimate the last term on the right-hand side of (3.40), we observe from (2.1), (3.7) and (3.11) that

$$\begin{cases} \|\rho^{1/2} u\|_{L^4}^4 \leq C \|\rho^{1/3} u\|_{L^3}^2 \|\nabla u\|_{L^2}^2 \leq C \|\nabla u\|_{L^2}^2 & \text{for } 0 \leq t \leq \sigma(T), \\ \|\rho^{1/2} u\|_{L^4}^4 \leq C \|\rho^{1/2} u\|_{L^2} \|\nabla u\|_{L^2}^3 \leq C \|\nabla u\|_{L^2}^2 & \text{for } \sigma(T) \leq t \leq T, \end{cases}$$

which, combined with (3.11), gives

$$\int_0^T \|\rho^{1/2} u\|_{L^4}^4 dt \leq C \int_0^T \|\nabla u\|_{L^2}^2 dt \leq C C_0.$$

This, together with (3.40) and (3.22), finishes the proof of (3.31).

To prove (3.32), operating $\sigma^m \dot{u}^j [\partial_t + \operatorname{div}(u \cdot)]$ with $m \geq 0$ to both sides of the j -th equation of (1.2) and integrating by parts over \mathbb{R}^3 , we obtain after summing up that

$$\begin{aligned} L &\triangleq \frac{1}{2} \frac{d}{dt} \int \sigma^m \rho |\dot{u}|^2 dx - \mu \int \sigma^m \dot{u}^j \left[\Delta u_t^j + \operatorname{div}(u \Delta u^j) \right] dx \\ &\quad - (\mu + \lambda) \int \sigma^m \dot{u}^j [\partial_j \operatorname{div} u_t + \operatorname{div}(u \partial_j \operatorname{div} u)] dx \\ &= \frac{m}{2} \sigma^{m-1} \sigma' \int \rho |\dot{u}|^2 dx - \int \sigma^m \dot{u}^j [\partial_j P_t + \operatorname{div}(u \partial_j P)] dx \\ &\quad + \int \sigma^m M(d)_t : \nabla \dot{u} dx + \int \sigma^m \operatorname{div} M(d) \cdot (u \cdot \nabla \dot{u}) dx \triangleq \sum_{i=1}^4 R_i. \end{aligned} \quad (3.41)$$

After integrating by parts and using Cauchy-Schwarz inequality, we easily see that

$$\begin{aligned} L_1 &\triangleq -\mu \int \sigma^m \dot{u}^j \left[\Delta u_t^j + \operatorname{div}(u \Delta u^j) \right] dx \\ &= \mu \int \sigma^m \left(|\nabla \dot{u}|^2 - \partial_k \dot{u}^j \partial_k (u \cdot \nabla u^j) + \partial_k \dot{u}^j u^k \Delta u^j \right) dx \end{aligned}$$

$$\begin{aligned}
&= \mu \int \sigma^m \left(|\nabla \dot{u}|^2 - \partial_k \dot{u}^j \partial_k u^l \partial_l u^j + \partial_k \dot{u}^j \partial_l u^l \partial_k u^j - \partial_k \dot{u}^j \partial_l u^k \partial_l u^j \right) dx \\
&\geq \frac{7\mu}{8} \sigma^m \|\nabla \dot{u}\|_{L^2}^2 - C \sigma^m \|\nabla u\|_{L^4}^4,
\end{aligned}$$

where and in what follows, we use the Einstein convention that repeated indices denote the summation with respect to those indices. Similarly,

$$\begin{aligned}
L_2 &\triangleq (\mu + \lambda) \int \sigma^m \dot{u}^j [\partial_j \operatorname{div} u_t + \operatorname{div}(u \partial_j \operatorname{div} u)] dx \\
&\geq (\mu + \lambda) \sigma^m \|\operatorname{div} \dot{u}\|_{L^2}^2 - \frac{\mu}{8} \sigma^m \|\nabla \dot{u}\|_{L^2}^2 - C \sigma^m \|\nabla u\|_{L^4}^4.
\end{aligned}$$

Thus, the left-hand side of (3.41) can be bounded from below as follows:

$$L \geq \frac{1}{2} \frac{d}{dt} \left(\sigma^m \|\rho^{1/2} \dot{u}\|_{L^2}^2 \right) + \frac{3\mu}{4} \sigma^m \|\nabla \dot{u}\|_{L^2}^2 - C \sigma^m \|\nabla u\|_{L^4}^4. \quad (3.42)$$

In view of (3.35), we have by integration by parts and (3.7) that

$$\begin{aligned}
R_2 &= \int \sigma^m \left[\partial_k \dot{u}^j u^k \partial_j P(\rho) - \operatorname{div} \dot{u} (\operatorname{div}(P(\rho)u) + (\gamma - 1)P(\rho)\operatorname{div} u) \right] dx \\
&= - \int \sigma^m \left(\partial_k \dot{u}^j \partial_j u^k P(\rho) + (\gamma - 1)P(\rho)(\operatorname{div} \dot{u})(\operatorname{div} u) \right) dx \\
&\leq \frac{\mu}{8} \sigma^m \|\nabla \dot{u}\|_{L^2}^2 + C \sigma^m \|\nabla u\|_{L^2}^2.
\end{aligned} \quad (3.43)$$

Using (2.1), (2.2) and (3.15), we get

$$\begin{aligned}
R_3 &\leq C \sigma^m \|\nabla d\|_{L^3} \|\nabla d_t\|_{L^6} \|\nabla \dot{u}\|_{L^2} \\
&\leq \frac{\mu}{8} \sigma^m \|\nabla \dot{u}\|_{L^2}^2 + C_1 C_0^{2\delta_0/3} \sigma^m \|\nabla^2 d_t\|_{L^2}^2,
\end{aligned} \quad (3.44)$$

and similarly,

$$\begin{aligned}
R_4 &\leq C \sigma^m \|\nabla d\|_{L^\infty} \|\nabla^2 d\|_{L^2}^{1/2} \|\nabla^2 d\|_{L^6}^{1/2} \|u\|_{L^6} \|\nabla \dot{u}\|_{L^2} \\
&\leq \frac{\mu}{8} \sigma^m \|\nabla \dot{u}\|_{L^2}^2 + C \sigma^m \|\nabla^2 d\|_{L^2}^2 \|\nabla^3 d\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \\
&\leq \frac{\mu}{8} \sigma^m \|\nabla \dot{u}\|_{L^2}^2 + C \sigma^m (\|\nabla u\|_{L^2}^4 + \|\nabla^2 d\|_{L^2}^4) \|\nabla^3 d\|_{L^2}^2.
\end{aligned} \quad (3.45)$$

Thus, putting (3.42)–(3.45) into (3.41), we know that

$$\begin{aligned}
&\frac{d}{dt} \left(\sigma^m \|\rho^{1/2} \dot{u}\|_{L^2}^2 \right) + \sigma^m \|\nabla \dot{u}\|_{L^2}^2 - C_1 C_0^{2\delta_0/3} \sigma^m \|\nabla^2 d_t\|_{L^2}^2 \\
&\leq C m \sigma^{m-1} \sigma' \|\rho^{1/2} \dot{u}\|_{L^2}^2 + C \sigma^m \|\nabla u\|_{L^2}^2 + C \sigma^m \|\nabla u\|_{L^4}^4 \\
&\quad + C \sigma^m (\|\nabla u\|_{L^2}^4 + \|\nabla^2 d\|_{L^2}^4) \|\nabla^3 d\|_{L^2}^2.
\end{aligned} \quad (3.46)$$

Next, we estimate $\|\nabla^2 d_t\|_{L^2}$. To do so, noticing that

$$d_{tt} - \Delta d_t = (|\nabla d|^2 d - u \cdot \nabla d)_t,$$

we obtain from direct computations that

$$\frac{d}{dt} \int \sigma^m |\nabla d_t|^2 dx + \int \sigma^m (|d_{tt}|^2 + |\nabla^2 d_t|^2) dx - m \sigma^{m-1} \sigma' \int |\nabla d_t|^2 dx$$

$$\begin{aligned}
&\leq C \int \sigma^m |\dot{u}|^2 |\nabla d|^2 dx + C \int \sigma^m |u|^2 |\nabla u|^2 |\nabla d|^2 dx + C \int \sigma^m |u|^2 |\nabla d_t|^2 dx \\
&\quad + C \int \sigma^m |\nabla d|^2 |\nabla d_t|^2 dx + C \int \sigma^m |d_t|^2 |\nabla d|^4 dx \triangleq \sum_{i=1}^5 J_i,
\end{aligned} \tag{3.47}$$

where we have used the facts that $|d| = 1$ and $u_t = \dot{u} - u \cdot \nabla u$.

For the first term on the right-hand side of (3.47), it follows from (2.1) and (3.15) that

$$J_1 \leq C \sigma^m \|\dot{u}\|_{L^6}^2 \|\nabla d\|_{L^3}^2 \leq C_2 C_0^{2\delta_0/3} \sigma^m \|\nabla \dot{u}\|_{L^2}^2. \tag{3.48}$$

Using (2.1), (2.6), (3.11) and (3.15), we see that

$$\begin{aligned}
J_2 &\leq C \sigma^m \|u\|_{L^6}^2 \|\nabla d\|_{L^6}^2 \|\nabla u\|_{L^6}^2 \\
&\leq C \sigma^m \|\nabla u\|_{L^2}^2 \|\nabla d\|_{L^6}^2 (\|\rho \dot{u}\|_{L^2}^2 + \|P(\rho) - P(\tilde{\rho})\|_{L^6}^2 + \|\nabla d\|_{L^3}^2 \|\nabla^2 d\|_{L^2}^2) \\
&\leq C \sigma^m \|\nabla u\|_{L^2}^2 \|\nabla^2 d\|_{L^2}^2 \left(\|\rho^{1/2} \dot{u}\|_{L^2}^2 + C_0^{1/3} + \|\nabla d\|_{L^3}^2 \|\nabla^2 d\|_{L^6}^2 \right) \\
&\leq C \sigma^m (\|\nabla u\|_{L^2}^4 + \|\nabla d\|_{L^2}^4) \left(\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 \right) + C \sigma^m \|\nabla u\|_{L^2}^2 \|\nabla^2 d\|_{L^2}^2,
\end{aligned} \tag{3.49}$$

and using (2.1) and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\sum_{i=3}^5 J_i &\leq C \sigma^m (\|u\|_{L^6}^2 + \|\nabla d\|_{L^6}^2) \|\nabla d_t\|_{L^3}^2 + C \sigma^m \|d_t\|_{L^6}^2 \|\nabla d\|_{L^6}^4 \\
&\leq C \sigma^m (\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2) \|\nabla d_t\|_{L^2} \|\nabla^2 d_t\|_{L^2} + C \sigma^m \|\nabla^2 d\|_{L^2}^4 \|\nabla d_t\|_{L^2}^2 \\
&\leq \frac{1}{4} \sigma^m \|\nabla^2 d_t\|_{L^2}^2 + C \sigma^m (\|\nabla u\|_{L^2}^4 + \|\nabla^2 d\|_{L^2}^4) \|\nabla d_t\|_{L^2}^2,
\end{aligned}$$

which, combining with (3.47)–(3.49) gives

$$\begin{aligned}
&\frac{d}{dt} (\sigma^m \|\nabla d_t\|_{L^2}^2) + \sigma^m (\|\nabla^2 d_t\|_{L^2}^2 + \|d_{tt}\|_{L^2}^2) - C_2 C_0^{2\delta_0/3} \sigma^m \|\nabla \dot{u}\|_{L^2}^2 \\
&\leq C m \sigma^{m-1} \sigma' \|\nabla d_t\|_{L^2}^2 + C \sigma^m \|\nabla u\|_{L^2}^2 \|\nabla^2 d\|_{L^2}^2 \\
&\quad + C \sigma^m (\|\nabla u\|_{L^2}^4 + \|\nabla^2 d\|_{L^2}^4) \left(\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 \right).
\end{aligned} \tag{3.50}$$

Thus if C_0 is chosen to be such that

$$C_0 \leq \varepsilon_4 \triangleq \min \left\{ \varepsilon_3, (2C_1)^{-3/(2\delta_0)}, (2C_2)^{-3/(2\delta_0)} \right\},$$

then, choosing $m = 2$ in (3.46) and (3.50), adding them together, and integrating the resulting inequality over $(0, T)$, by (3.7) and (3.11) we obtain

$$\begin{aligned}
&\sup_{0 \leq t \leq T} \sigma^2 \left(\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 \right) \\
&\quad + \int_0^T \sigma^2 (\|\nabla \dot{u}\|_{L^2}^2 + \|d_{tt}\|_{L^2}^2 + \|\nabla^2 d_t\|_{L^2}^2) dt \\
&\leq C \int_0^T \sigma^2 \|\nabla u\|_{L^2}^2 dt + C \int_0^T \sigma^2 \|\nabla u\|_{L^2}^2 \|\nabla^2 d\|_{L^2}^2 dt
\end{aligned}$$

$$\begin{aligned}
& +C \int_0^T \sigma^2 (\|\nabla u\|_{L^2}^4 + \|\nabla^2 d\|_{L^2}^4) \left(\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 \right) dt \\
& +C \int_0^T \sigma \left(\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 \right) dt + C \int_0^T \sigma^2 \|\nabla u\|_{L^4}^4 dt \\
& \leq CC_0^{1/2+2\delta_0} + CA_1(T) + C \int_0^T \sigma^2 \|\nabla u\|_{L^4}^4 dt,
\end{aligned} \tag{3.51}$$

since it follows from (3.7) and (3.12) that

$$\begin{aligned}
& \int_0^T \sigma^2 (\|\nabla u\|_{L^2}^4 + \|\nabla^2 d\|_{L^2}^4) \left(\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 \right) dt \\
& \leq \sup_{0 \leq t \leq T} \sigma^2 \left(\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 \right) \int_0^T (\|\nabla u\|_{L^2}^4 + \|\nabla^2 d\|_{L^2}^4) dt \\
& \leq CC_0^{2\delta_0} A_2(T) \leq CC_0^{1/2+2\delta_0}.
\end{aligned}$$

Moreover, it follows from (3.16) and the standard L^2 -estimate of elliptic system that

$$\begin{aligned}
\|\nabla^3 d\|_{L^2} & \leq C (\|\nabla d_t\|_{L^2} + \|\nabla d\|_{L^6}^3 + \|\nabla d \nabla^2 d\|_{L^2}) \\
& +C (\|\nabla u \nabla d\|_{L^2} + \|u \nabla^2 d\|_{L^2}) \\
& \leq C \left(\|\nabla d_t\|_{L^2} + \|\nabla^2 d\|_{L^2}^3 + \|\nabla^2 d\|_{L^2}^{3/2} \|\nabla^3 d\|_{L^2}^{1/2} \right) \\
& +C \|\nabla u\|_{L^2} \|\nabla^2 d\|_{L^2}^{1/2} \|\nabla^3 d\|_{L^2}^{1/2},
\end{aligned}$$

so that,

$$\|\nabla^3 d\|_{L^2}^2 \leq C \|\nabla^2 d\|_{L^2}^2 (\|\nabla^2 d\|_{L^2}^4 + \|\nabla u\|_{L^2}^4) + C \|\nabla d_t\|_{L^2}^2. \tag{3.52}$$

Therefore, keeping in mind that $\beta > 1/2$, we infer from (3.7) and (3.51) that

$$\begin{aligned}
& \sup_{0 \leq t \leq T} (\sigma^2 \|\nabla^3 d\|_{L^2}^2) \\
& \leq C \sup_{0 \leq t \leq T} (\sigma^2 \|\nabla d_t\|_{L^2}^2) + C \sup_{0 \leq t \leq T} (\sigma^2 \|\nabla^2 d\|_{L^2}^2 (\|\nabla u\|_{L^2}^4 + \|\nabla^2 d\|_{L^2}^4)) \\
& \leq CC_0^{1/2+2\delta_0} + CA_1(T) + C \int_0^T \sigma^2 \|\nabla u\|_{L^4}^4 dt \\
& +C \sup_{0 \leq t \leq \sigma(T)} (\sigma \|\nabla^2 d\|_{L^2}^2) \sup_{0 \leq t \leq \sigma(T)} \left(\sigma^{(3-2\beta)/4} (\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2) \right)^2 \\
& +C \sup_{\sigma(T) \leq t \leq T} (\sigma \|\nabla^2 d\|_{L^2}^2) \sup_{\sigma(T) \leq t \leq T} (\sigma (\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2))^2 \\
& \leq CC_0^{1/2+2\delta_0} + CA_1(T) + C \int_0^T \sigma^2 \|\nabla u\|_{L^4}^4 dt,
\end{aligned}$$

which, together with (3.51), finishes the proof of (3.32). \square

The next lemma plays an important role in the proof of the uniform upper bound of ρ . The ideas of the proof are motivated by the ones in [15, 17].

Lemma 3.5 *Let (ρ, u, d) be a smooth solution of (1.1)–(1.7) on $\mathbb{R}^3 \times (0, T]$ satisfying (3.7). Then there exist positive constants ε_5 and C depending on $\mu, \lambda, A, \gamma, \bar{\rho}, \tilde{\rho}, \beta, M_1$ and M_2 , such that*

$$\sup_{0 \leq t \leq \sigma(T)} \left(t^{1-\beta} \|\nabla u\|_{L^2}^2 \right) + \int_0^{\sigma(T)} t^{1-\beta} \|\rho^{1/2} \dot{u}\|_{L^2}^2 dt \leq C(M_1, M_2), \quad (3.53)$$

and

$$\begin{aligned} & \sup_{0 \leq t \leq \sigma(T)} t^{2-\beta} \left(\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 \right) \\ & + \int_0^{\sigma(T)} t^{2-\beta} \left(\|\nabla \dot{u}\|_{L^2}^2 + \|d_{tt}\|_{L^2}^2 + \|\nabla^2 d_t\|_{L^2}^2 \right) dt \leq C(M_1, M_2), \end{aligned} \quad (3.54)$$

provided $C_0 \leq \varepsilon_5$.

Proof. The proof of (3.53) is motivated by [15, 17]. For a fixed smooth solution (ρ, u, d) , we define the linear differential operator \mathcal{L} acting on the functions $w : \mathbb{R}^3 \times [0, \infty) \rightarrow \mathbb{R}^3$ by

$$\begin{aligned} (\mathcal{L}w)^j & \triangleq \rho w_t^j + \rho u \cdot \nabla w^j - (\mu \Delta w^j + (\mu + \lambda) \operatorname{div} w_{x_j}) \\ & = \rho \dot{w}^j - (\mu \Delta w^j + (\mu + \lambda) \operatorname{div} w_{x_j}), \quad j = 1, 2, 3, \end{aligned} \quad (3.55)$$

where $\dot{w} \triangleq w_t + u \cdot \nabla w$. We thus define w_1, w_2 and w_3 by

$$\mathcal{L}w_1 = 0, \quad w_1(x, 0) = w_{10}(x), \quad (3.56)$$

$$\mathcal{L}w_2 = -\nabla P(\rho), \quad w_2(x, 0) = 0, \quad (3.57)$$

and

$$\mathcal{L}w_3 = -\operatorname{div}(\nabla d \odot \nabla v - \frac{1}{2} \nabla d : \nabla v \mathbb{I}_3), \quad w_3(x, 0) = 0, \quad (3.58)$$

where $v = v(x, t)$ is the solution of (3.23).

The estimates of w_1, w_2 are similar to those in [15, 17]. For the reader's convenience, we reproduce the proofs here. Multiplying (3.56) and (3.57) by w_1 and w_2 , respectively, and integrating them by parts over $\mathbb{R}^3 \times (0, \sigma(T))$, one easily deduces from (3.7) and (3.11) that

$$\sup_{0 \leq t \leq \sigma(T)} \int \rho |w_1|^2 dx + \int_0^{\sigma(T)} \|\nabla w_1\|_{L^2}^2 dt \leq C \|w_{10}\|_{L^2}^2, \quad (3.59)$$

and

$$\sup_{0 \leq t \leq \sigma(T)} \int \rho |w_2|^2 dx + \int_0^{\sigma(T)} \|\nabla w_2\|_{L^2}^2 dt \leq C C_0. \quad (3.60)$$

To estimate w_3 , multiplying (3.58) by w_3 and integrating it by parts over $\mathbb{R}^3 \times (0, \sigma(T))$, by (2.1) and (3.15) we get

$$\begin{aligned} & \sup_{0 \leq t \leq \sigma(T)} \int \rho |w_3|^2 dx + \int_0^{\sigma(T)} \|\nabla w_3\|_{L^2}^2 dt \\ & \leq C \int_0^{\sigma(T)} \|\nabla d\|_{L^3} \|\nabla v\|_{L^6} \|\nabla w_3\|_{L^2} dt \end{aligned}$$

$$\leq \frac{1}{2} \int_0^{\sigma(T)} \|\nabla w_3\|_{L^2}^2 dt + C \int_0^{\sigma(T)} \|\nabla^2 v\|_{L^2}^2 dt,$$

which, combined with (3.24), yields

$$\sup_{0 \leq t \leq \sigma(T)} \int \rho |w_3|^2 dx + \int_0^{\sigma(T)} \|\nabla w_3\|_{L^2}^2 dt \leq C \|\nabla v_0\|_{L^2}^2. \quad (3.61)$$

Next we estimate $\|\nabla w_i\|_{L^2}$ ($i = 1, 2, 3$). Thanks to (3.7), an application of the standard L^2 -estimate of elliptic system to (3.56) shows

$$\|\nabla w_1\|_{L^6} \leq C \|\nabla^2 w_1\|_{L^2} \leq C \|\rho^{1/2} \dot{w}_1\|_{L^2}.$$

So, multiplying (3.56) by w_{1t} in L^2 and integrating by parts, we infer from (3.7) that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int (\mu |\nabla w_1|^2 + (\mu + \lambda)(\operatorname{div} w_1)^2) dx + \int \rho |\dot{w}_1|^2 dx \\ &= \int \rho \dot{w}_1 \cdot (u \cdot \nabla w_1) dx \\ &\leq C \left(\int \rho |\dot{w}_1|^2 dx \right)^{1/2} \left(\int \rho |u|^3 dx \right)^{1/3} \|\nabla w_1\|_{L^6} \\ &\leq C \left(\int \rho |u|^3 dx \right)^{1/3} \int \rho |\dot{w}_1|^2 dx \\ &\leq C_1 C_0^{\delta_0/3} \int \rho |\dot{w}_1|^2 dx \end{aligned} \quad (3.62)$$

for any $t \in (0, \sigma(T))$. Thus if C_0 is chosen to be such that

$$C_0 \leq \varepsilon_{5,1} \triangleq \min \left\{ \varepsilon_4, (2C_1)^{-3/\delta_0} \right\},$$

then it follows from (3.62) that

$$\sup_{0 \leq t \leq \sigma(T)} \|\nabla w_1\|_{L^2}^2 + \int_0^{\sigma(T)} \|\rho^{1/2} \dot{w}_1\|_{L^2}^2 dt \leq C \|\nabla w_{10}\|_{L^2}^2. \quad (3.63)$$

On the other hand, multiplying (3.62) by t and integrating it over $(0, \sigma(T))$, by (3.59) we see that if $C_0 \leq \varepsilon_{5,1}$, then

$$\begin{aligned} & \sup_{0 \leq t \leq \sigma(T)} (t \|\nabla w_1\|_{L^2}^2) + \int_0^{\sigma(T)} t \int \rho |\dot{w}_1|^2 dx dt \\ &\leq C \int_0^{\sigma(T)} \|\nabla w_1\|_{L^2}^2 dt \leq C \|w_{10}\|_{L^2}^2. \end{aligned} \quad (3.64)$$

Since the solution operator $w_{10} \mapsto w_1(\cdot, t)$ is linear, by the standard Riesz-Thorin interpolation argument (see [2]), we conclude from (3.63) and (3.64) that for $\beta \in (1/2, 1]$,

$$\sup_{0 \leq t \leq \sigma(T)} \left(t^{1-\beta} \|\nabla w_1\|_{L^2}^2 \right) + \int_0^{\sigma(T)} t^{1-\beta} \|\rho^{1/2} \dot{w}_1\|_{L^2}^2 dt \leq C \|w_{10}\|_{\dot{H}^\beta}^2. \quad (3.65)$$

In order to estimate $\|\nabla w_2\|_{L^2}$, analogous to the proof of Lemma 2.2, we set

$$\tilde{F} \triangleq (2\mu + \lambda)\operatorname{div} w_2 - (P(\rho) - P(\tilde{\rho})).$$

Then it follows from (3.57) that

$$\|\nabla \tilde{F}\|_{L^2} \leq C\|\rho \dot{w}_2\|_{L^2}, \quad \|\nabla w_2\|_{L^6} \leq C(\|\rho \dot{w}_2\|_{L^2} + \|P(\rho) - P(\tilde{\rho})\|_{L^6}). \quad (3.66)$$

Now, multiplying (3.59) by w_{2t} in L^2 and integrating by parts, by (3.35) we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int (\mu |\nabla w_2|^2 + (\mu + \lambda)(\operatorname{div} w_2)^2) dx + \int \rho |\dot{w}_2|^2 dx \\ &= \frac{d}{dt} \int (P(\rho) - P(\tilde{\rho})) \operatorname{div} w_2 dx - \int (P(\rho) - P(\tilde{\rho})) u \cdot \nabla \operatorname{div} w_2 dx \\ & \quad + \int ((\gamma - 1)P(\rho) + P(\tilde{\rho})) (\operatorname{div} u)(\operatorname{div} w_2) dx + \int \rho \dot{w}_2 \cdot (u \cdot \nabla w_2) dx \\ & \triangleq \frac{d}{dt} I_0 + \sum_{i=1}^3 I_i. \end{aligned} \quad (3.67)$$

Since $(2\mu + \lambda)\operatorname{div} w_2 = \tilde{F} + (P(\rho) - P(\tilde{\rho}))$, integrating by parts and using (2.1), (3.7), (3.11) and (3.66), we find that

$$\begin{aligned} I_1 + I_2 &\leq C(\|P(\rho) - P(\tilde{\rho})\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla w_2\|_{L^2}^2) \\ & \quad + C\|P(\rho) - P(\tilde{\rho})\|_{L^3} \|u\|_{L^6} \|\nabla \tilde{F}\|_{L^2} \\ &\leq \frac{1}{4} \|\rho^{1/2} \dot{w}_2\|_{L^2}^2 + C(\|\nabla w_2\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + C_0), \end{aligned} \quad (3.68)$$

and using (3.7), (3.11) and (3.66), one easily gets

$$\begin{aligned} I_3 &\leq C \left(\int \rho |\dot{w}_2|^2 dx \right)^{1/2} \left(\int \rho |u|^3 dx \right)^{1/3} \|\nabla w_2\|_{L^6} \\ &\leq CC_0^{\delta_0/3} \|\rho^{1/2} \dot{w}_2\|_{L^2} \left(\|\rho^{1/2} \dot{w}_2\|_{L^2} + \|P(\rho) - P(\tilde{\rho})\|_{L^6} \right) \\ &\leq C_2 C_0^{\delta_0/3} \|\rho^{1/2} \dot{w}_2\|_{L^2}^2 + CC_0^{1/3}. \end{aligned} \quad (3.69)$$

Thus if C_0 is chosen to be such that

$$C_0 \leq \varepsilon_{5,2} \triangleq \min \left\{ \varepsilon_{5,1}, (4C_2)^{-3/\delta_0} \right\},$$

then we obtain after putting (3.68), (3.69) into (3.67) and integrating it over $(0, \sigma(T))$ that

$$\begin{aligned} & \sup_{0 \leq t \leq \sigma(T)} \|\nabla w_2\|_{L^2}^2 + \int_0^{\sigma(T)} \|\rho^{1/2} \dot{w}_2\|_{L^2}^2 dt \\ & \leq CC_0^{1/3} + C \int_0^{\sigma(T)} (\|\nabla w_2\|_{L^2}^2 + \|\nabla u\|_{L^2}^2) dt \leq CC_0^{1/3}, \end{aligned} \quad (3.70)$$

where we have also used (3.11), (3.60) and the following simple fact (due to (3.11)) :

$$|I_0| \leq \frac{\mu}{4} \|\nabla w_2\|_{L^2}^2 + C\|P(\rho) - P(\tilde{\rho})\|_{L^2}^2 \leq \frac{\mu}{4} \|\nabla w_2\|_{L^2}^2 + CC_0.$$

We now estimate $\|\nabla w_3\|_{L^2}$. To do so, using (2.1), (3.7), (3.15) and the standard H^2 -regularity of elliptic system, we first deduce from (3.58) that

$$\begin{aligned} & \|\nabla w_3\|_{L^6} + \|\nabla^2 w_3\|_{L^2} \\ & \leq C (\|\rho \dot{w}_3\|_{L^2} + \|\nabla d \nabla^2 v\|_{L^2} + \|\nabla v \nabla^2 d\|_{L^2}) \\ & \leq C \left(\|\rho^{1/2} \dot{w}_3\|_{L^2} + C_0^{\delta_0/3} \|\nabla^3 v\|_{L^2} + \|\nabla^2 v\|_{L^2}^{1/2} \|\nabla^3 v\|_{L^2}^{1/2} \|\nabla^2 d\|_{L^2} \right). \end{aligned} \quad (3.71)$$

Then, multiplying (3.58) by w_{3t} in L^2 and integrating by parts, we find

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int (\mu |\nabla w_3|^2 + (\mu + \lambda) (\operatorname{div} w_3)^2) dx + \int \rho |\dot{w}_3|^2 dx \\ & = \frac{d}{dt} \int \left(\nabla d \odot \nabla v - \frac{1}{2} \nabla d : \nabla v \mathbb{I}_3 \right) : \nabla w_3 dx \\ & \quad - \int \left(\nabla d \odot \nabla v_t - \frac{1}{2} \nabla d : \nabla v_t \mathbb{I}_3 \right) : \nabla w_3 dx \\ & \quad - \int \left(\nabla d_t \odot \nabla v - \frac{1}{2} \nabla d_t : \nabla v \mathbb{I}_3 \right) : \nabla w_3 dx + \int \rho \dot{w}_3 \cdot (u \cdot \nabla w_3) dx \\ & = \frac{d}{dt} J_0 + \sum_{i=1}^3 J_i. \end{aligned} \quad (3.72)$$

It readily follows from (2.1) and (3.15) that

$$|J_0| \leq C \|\nabla d\|_{L^3} \|\nabla v\|_{L^6} \|\nabla w_3\|_{L^2} \leq \frac{\mu}{4} \|\nabla w_3\|_{L^2}^2 + C C_0^{2\delta_0/3} \|\nabla^2 v\|_{L^2}^2. \quad (3.73)$$

By (3.15), (3.71) and Cauchy-Schwarz inequality, we have

$$\begin{aligned} J_1 & \leq C \|\nabla d\|_{L^3} \|\nabla v_t\|_{L^2} \|\nabla w_3\|_{L^6} \\ & \leq C C_0^{\delta_0/3} \|\nabla v_t\|_{L^2} \left(\|\rho^{1/2} \dot{w}_3\|_{L^2} + C_0^{\delta_0/3} \|\nabla^3 v\|_{L^2} + \|\nabla^2 v\|_{L^2}^{1/2} \|\nabla^3 v\|_{L^2}^{1/2} \|\nabla^2 d\|_{L^2} \right) \\ & \leq \frac{1}{4} \|\rho^{1/2} \dot{w}_3\|_{L^2}^2 + C (\|\nabla v_t\|_{L^2}^2 + \|\nabla^3 v\|_{L^2}^2 + \|\nabla^2 v\|_{L^2}^2 \|\nabla^2 d\|_{L^2}^4), \end{aligned} \quad (3.74)$$

and similarly, after integrating by parts and using Cauchy-Schwarz inequality, we have

$$\begin{aligned} J_2 & \leq C \int (|d_t| |\nabla^2 v| |\nabla w_3| + |d_t| |\nabla v| |\nabla^2 w_3|) dx \\ & \leq C (\|d_t\|_{L^2} \|\nabla^2 v\|_{L^3} \|\nabla w_3\|_{L^6} + \|d_t\|_{L^2} \|\nabla v\|_{L^\infty} \|\nabla^2 w_3\|_{L^2}) \\ & \leq C \|d_t\|_{L^2} \|\nabla^2 v\|_{L^2}^{1/2} \|\nabla^3 v\|_{L^2}^{1/2} \|\nabla^2 w_3\|_{L^2} \\ & \leq C \|d_t\|_{L^2} \|\nabla^2 v\|_{L^2}^{1/2} \|\nabla^3 v\|_{L^2}^{1/2} \\ & \quad \times \left(\|\rho^{1/2} \dot{w}_3\|_{L^2} + C_0^{\delta_0/3} \|\nabla^3 v\|_{L^2} + \|\nabla^2 v\|_{L^2}^{1/2} \|\nabla^3 v\|_{L^2}^{1/2} \|\nabla^2 d\|_{L^2} \right) \\ & \leq \frac{1}{4} \|\rho^{1/2} \dot{w}_3\|_{L^2}^2 + C (\|\nabla^3 v\|_{L^2}^2 + \|\nabla^2 v\|_{L^2}^2 \|\nabla^2 d\|_{L^2}^4 + \|\nabla^2 v\|_{L^2}^2 \|d_t\|_{L^2}^4). \end{aligned} \quad (3.75)$$

Finally, similar to that in (3.72), we infer from (3.7), (3.15) and (3.71) that

$$J_3 \leq C \left(\int \rho |\dot{w}_3|^2 dx \right)^{1/2} \left(\int \rho |u|^3 dx \right)^{1/3} \|\nabla w_3\|_{L^6}$$

$$\begin{aligned}
&\leq CC_0^{\delta_0/3} \|\rho^{1/2} \dot{w}_3\|_{L^2} \left(\|\rho^{1/2} \dot{w}_3\|_{L^2} + C_0^{\delta_0/3} \|\nabla^3 v\|_{L^2} + \|\nabla^2 v\|_{L^2}^{1/2} \|\nabla^3 v\|_{L^2}^{1/2} \|\nabla^2 d\|_{L^2} \right) \\
&\leq C_3 C_0^{\delta_0/3} \|\rho^{1/2} \dot{w}_3\|_{L^2}^2 + C \left(\|\nabla^3 v\|_{L^2}^2 + \|\nabla^2 v\|_{L^2}^2 \|\nabla^2 d\|_{L^2}^4 \right).
\end{aligned} \tag{3.76}$$

Thus if C_0 is chosen to be such that

$$C_0 \leq \varepsilon_5 \triangleq \min \left\{ \varepsilon_{5,2}, (4C_3)^{-3/\delta_0} \right\},$$

then by (3.12) and (3.28) we deduce after putting (3.73)–(3.76) into (3.72) that

$$\begin{aligned}
&\sup_{0 \leq t \leq \sigma(T)} \|\nabla w_3\|_{L^2}^2 + \int_0^{\sigma(T)} \|\rho^{1/2} \dot{w}_3\|_{L^2}^2 dt \\
&\leq C \sup_{0 \leq t \leq \sigma(T)} \|\nabla^2 v\|_{L^2}^2 + C \int_0^{\sigma(T)} (\|\nabla v_t\|_{L^2}^2 + \|\nabla^3 v\|_{L^2}^2) dt \\
&\quad + C \sup_{0 \leq t \leq \sigma(T)} \|\nabla^2 v\|_{L^2}^2 \int_0^{\sigma(T)} (\|d_t\|_{L^2}^4 + \|\nabla^2 d\|_{L^2}^4) dt \\
&\leq C \|\nabla^2 v_0\|_{L^2}^2,
\end{aligned} \tag{3.77}$$

where we have used (1.3), (3.12) and (3.15) to get that

$$\begin{aligned}
\int_0^T \|d_t\|_{L^2}^4 dt &\leq C \int_0^T (\|\nabla^2 d\|_{L^2} + \|u \nabla d\|_{L^2} + \|\nabla d\|_{L^4}^2)^4 dt \\
&\leq C \int_0^T (\|\nabla^2 d\|_{L^2} + \|\nabla u\|_{L^2} \|\nabla d\|_{L^3} + \|\nabla d\|_{L^3} \|\nabla^2 d\|_{L^2})^4 dt \\
&\leq C \int_0^T (\|\nabla^2 d\|_{L^2}^4 + \|\nabla u\|_{L^2}^4) dt \leq CC_0^{2\delta_0}.
\end{aligned}$$

Similarly, multiplying (3.72) by t , integrating it over $(0, \sigma(T))$, and taking (3.73)–(3.76) into account, we deduce from (3.29) and (3.61) that

$$\begin{aligned}
&\sup_{0 \leq t \leq \sigma(T)} (t \|\nabla w_3\|_{L^2}^2) + \int_0^{\sigma(T)} t \|\rho^{1/2} \dot{w}_3\|_{L^2}^2 dt \\
&\leq C \sup_{0 \leq t \leq \sigma(T)} (t \|\nabla^2 v\|_{L^2}^2) + C \int_0^{\sigma(T)} t (\|\nabla v_t\|_{L^2}^2 + \|\nabla^3 v\|_{L^2}^2) dt \\
&\quad + C \sup_{0 \leq t \leq \sigma(T)} (t \|\nabla^2 v\|_{L^2}^2) \int_0^{\sigma(T)} (\|d_t\|_{L^2}^4 + \|\nabla^2 d\|_{L^2}^4) dt + C \int_0^{\sigma(T)} \|\nabla w_3\|_{L^2}^2 dt \\
&\leq C \|\nabla v_0\|_{L^2}^2.
\end{aligned} \tag{3.78}$$

By the same token as that in the proof of (3.65), we conclude from (3.77) and (3.78) that

$$\sup_{0 \leq t \leq \sigma(T)} \left(t^{1-\beta} \|\nabla w_3\|_{L^2}^2 \right) + \int_0^{\sigma(T)} t^{1-\beta} \|\rho^{1/2} \dot{w}_3\|_{L^2}^2 dt \leq C \|\nabla v_0\|_{\dot{H}^\beta}^2. \tag{3.79}$$

Now, choosing $w_{10} = u_0$ and $v_0 = d_0$ so that $w_1 + w_2 + w_3 = u$ and $v = d$, we immediately obtain (3.53) from (3.65), (3.70) and (3.79).

To prove (3.54), choosing $m = 2 - \beta$ in (3.46), (3.50) and adding them together, we infer from (3.7), (3.11), (3.21) and (3.53) that if $C_0 \leq \varepsilon_5$, then

$$\begin{aligned}
& \sup_{0 \leq t \leq \sigma(T)} t^{2-\beta} \left(\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 \right) \\
& + \int_0^{\sigma(T)} t^{2-\beta} (\|\nabla \dot{u}\|_{L^2}^2 + \|d_{tt}\|_{L^2}^2 + \|\nabla^2 d_t\|_{L^2}^2) dt \\
& \leq C \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 dt + C \int_0^{\sigma(T)} t^{2-\beta} \|\nabla u\|_{L^2}^2 \|\nabla^2 d\|_{L^2}^2 dt \\
& + C \int_0^{\sigma(T)} t^{2-\beta} (\|\nabla u\|_{L^2}^4 + \|\nabla d\|_{L^2}^4) \left(\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 \right) dt \\
& + C \int_0^{\sigma(T)} t^{1-\beta} \left(\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 \right) dt + C \int_0^{\sigma(T)} t^{2-\beta} \|\nabla u\|_{L^4}^4 dt \\
& \leq C + C \int_0^{\sigma(T)} t^{2-\beta} \|\nabla u\|_{L^4}^4 dt, \tag{3.80}
\end{aligned}$$

since it follows from (3.21) and (3.53) that for $\beta \in (1/2, 1]$,

$$\begin{aligned}
& \int_0^{\sigma(T)} t^{2-\beta} (\|\nabla u\|_{L^2}^4 + \|\nabla d\|_{L^2}^4) \left(\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 \right) dt \\
& \leq \sup_{0 \leq t \leq \sigma(T)} \left(t^{1-\beta} (\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2) \right)^2 \\
& \quad \times \int_0^{\sigma(T)} t^{1-\beta} \left(\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 \right) dt \\
& \leq C.
\end{aligned}$$

Using Lemmas 2.1, 2.2, (3.7), (3.11), (3.12), (3.21) and (3.53), we have

$$\begin{aligned}
& \int_0^{\sigma(T)} t^{2-\beta} \|\nabla u\|_{L^4}^4 dt \leq C \int_0^{\sigma(T)} t^{2-\beta} \|\nabla u\|_{L^2} \|\nabla u\|_{L^6}^3 dt \\
& \leq C \int_0^{\sigma(T)} t^{2-\beta} \|\nabla u\|_{L^2} \left(1 + \|\rho^{1/2} \dot{u}\|_{L^2}^3 + \|\nabla^2 d\|_{L^2}^{9/2} \|\nabla^3 d\|_{L^2}^{3/2} \right) dt \\
& \leq C + C \int_0^{\sigma(T)} t^{\beta-1/2} \left(t^{1-\beta} \|\nabla u\|_{L^2}^2 \right)^{1/2} \left(t^{2-\beta} \|\rho^{1/2} \dot{u}\|_{L^2}^2 \right)^{1/2} \left(t^{1-\beta} \|\rho^{1/2} \dot{u}\|_{L^2}^2 \right) dt \\
& \quad + C \int_0^{\sigma(T)} t^{2\beta-1} (\|\nabla u\|_{L^2}^4)^{1/4} \left(t^{1-\beta} \|\nabla^2 d\|_{L^2}^2 \right)^{9/4} \left(t^{1-\beta} \|\nabla^3 d\|_{L^2}^2 \right)^{3/4} dt \\
& \leq C + C \sup_{0 \leq t \leq \sigma(T)} \left(\left(t^{1-\beta} \|\nabla u\|_{L^2}^2 \right)^{1/2} \left(t^{2-\beta} \|\rho^{1/2} \dot{u}\|_{L^2}^2 \right)^{1/2} \right) \int_0^{\sigma(T)} t^{1-\beta} \|\rho^{1/2} \dot{u}\|_{L^2}^2 dt \\
& \quad + C \sup_{0 \leq t \leq \sigma(T)} \left(t^{1-\beta} \|\nabla^2 d\|_{L^2}^2 \right)^{9/4} \left(\int_0^{\sigma(T)} \|\nabla u\|_{L^2}^4 dt \right)^{1/4} \left(\int_0^{\sigma(T)} t^{1-\beta} \|\nabla^3 d\|_{L^2}^2 dt \right)^{3/4} \\
& \leq C + C \sup_{0 \leq t \leq \sigma(T)} \left(t^{2-\beta} \|\rho^{1/2} \dot{u}\|_{L^2}^2 \right)^{1/2}, \tag{3.81}
\end{aligned}$$

provided $C_0 \leq \varepsilon_5$. As a result, putting (3.81) into (3.80) and using Young's inequality, we immediately obtain (3.54). The proof of Lemma 3.6 is therefore completed. \square

We are now in a position of estimating $A_5(\sigma(T))$.

Lemma 3.6 *Let (ρ, u, d) be a smooth solution of (1.1)–(1.7) on $\mathbb{R}^3 \times (0, T]$ satisfying (3.7). Then there exists a constant $\varepsilon_6 > 0$, depending only on $\mu, \lambda, A, \gamma, \bar{\rho}, \tilde{\rho}, \beta, M_1$ and M_2 , such that*

$$A_5(\sigma(T)) \triangleq \sup_{0 \leq t \leq \sigma(T)} \int \rho |u|^3 dx \leq \frac{C_0^{\delta_0}}{2}, \quad (3.82)$$

provided $C_0 \leq \varepsilon_6$.

Proof. Multiplying (1.2) by $3|u|u$ and integrating it by parts over $\mathbb{R}^3 \times (0, \sigma(T))$, we have from Cauchy-Schwarz inequality that

$$\begin{aligned} \sup_{0 \leq t \leq \sigma(T)} \int \rho |u|^3 dx &\leq \int \rho_0 |u_0|^3 dx + C \int_0^{\sigma(T)} \int |P(\rho) - P(\tilde{\rho})| |u| |\nabla u| dx dt \\ &\quad + C \int_0^{\sigma(T)} \int |u| |\nabla u|^2 dx dt + C \int_0^{\sigma(T)} \int |\nabla d|^4 |u| dx dt \\ &\triangleq \int \rho_0 |u_0|^3 dx + \sum_{i=1}^3 R_i. \end{aligned} \quad (3.83)$$

First, it is easily seen from (2.1), (3.7) and (3.11) that

$$R_1 \leq C \int_0^{\sigma(T)} \|P(\rho) - P(\tilde{\rho})\|_{L^3} \|u\|_{L^6} \|\nabla u\|_{L^2} dt \leq C \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 dt \leq CC_0. \quad (3.84)$$

Keeping in mind that $\beta \in (1/2, 1]$ and $\delta_0 \in (0, 1/9]$, using Lemmas 2.1, 2.2, (3.7), (3.11), (3.12), (3.15) and Hölder inequality, we deduce

$$\begin{aligned} R_2 &\leq C \int_0^{\sigma(T)} \|u\|_{L^\infty} \|\nabla u\|_{L^2}^2 dt \leq C \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^{5/2} \|\nabla u\|_{L^6}^{1/2} dt \\ &\leq C \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^{5/2} (\|\rho \dot{u}\|_{L^2} + \|\nabla d \nabla^2 d\|_{L^2} + \|P(\rho) - P(\tilde{\rho})\|_{L^6})^{1/2} dt \\ &\leq C \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^{5/2} \left(\|\rho^{1/2} \dot{u}\|_{L^2} + \|\nabla d\|_{L^3} \|\nabla^3 d\|_{L^2} + C_0^{1/6} \right)^{1/2} dt \\ &\leq C \int_0^{\sigma(T)} \sigma^{3(2\beta-3)/8} \left(\sigma^{(3-2\beta)/4} \|\nabla u\|_{L^2}^2 \right)^{5/4} \left(\sigma^{(3-2\beta)/4} \left(\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 \right) \right)^{1/4} dt \\ &\quad + CC_0^{1/12} \left(\int_0^{\sigma(T)} \|\nabla u\|_{L^2}^4 dt \right)^{5/8} \\ &\leq CC_0^{3\delta_0/2} + C \sup_{0 \leq t \leq \sigma(T)} \left(\sigma^{(3-2\beta)/4} \|\nabla u\|_{L^2}^2 \right)^{5/4} \\ &\quad \times \left(\int_0^{\sigma(T)} \sigma^{(2\beta-3)/2} dt \right)^{3/4} \left(\int_0^{\sigma(T)} \sigma^{(3-2\beta)/4} \left(\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 \right) dt \right)^{1/4} \\ &\leq CC_0^{3\delta_0/2}, \end{aligned} \quad (3.85)$$

and similarly,

$$\begin{aligned}
R_3 &\leq C \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^{1/2} \|\nabla u\|_{L^6}^{1/2} \|\nabla d\|_{L^3}^2 \|\nabla^2 d\|_{L^2}^2 dt \\
&\leq C C_0^{2\delta_0/3} \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^{1/2} \|\nabla^2 d\|_{L^2}^2 \left(C_0^{1/6} + \|\rho^{1/2} \dot{u}\|_{L^2} + \|\nabla^3 d\|_{L^2} \right)^{1/2} dt \\
&\leq C C_0^{1/12} \left(\int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 dt \right)^{1/4} \left(\int_0^{\sigma(T)} \|\nabla^2 d\|_{L^2}^4 dt \right)^{1/2} \\
&\quad + C \sup_{0 \leq t \leq \sigma(T)} \left(\left(\sigma^{(3-2\beta)/4} \|\nabla u\|_{L^2}^2 \right)^{1/4} \left(\sigma^{(3-2\beta)/4} \|\nabla^2 d\|_{L^2}^2 \right) \right) \\
&\quad \times \int_0^{\sigma(T)} \sigma^{3(2\beta-3)/8} \left(\sigma^{(3-2\beta)/4} \left(\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 \right) \right)^{1/4} dt \\
&\leq C C_0^{3\delta_0/2}.
\end{aligned}$$

This, combined with (3.84), (3.85) and (3.83) gives

$$\begin{aligned}
\sup_{0 \leq t \leq \sigma(T)} \int \rho |u|^3 dx &\leq \int \rho_0 |u_0|^3 dx + C C_0^{3\delta_0/2} \\
&\leq C C_0^{3(2\beta-1)/(4\beta)} + C C_0^{3\delta_0/2} \leq C_1 C_0^{3\delta_0/2},
\end{aligned} \tag{3.86}$$

where we have used the interpolation and Sobolev embedding inequalities to get that

$$\begin{aligned}
\left(\int \rho_0 |u_0|^3 dx \right)^{1/3} &\leq C \left(\int \rho_0 |u_0|^2 dx \right)^{(2\beta-1)/(4\beta)} \left(\int |u_0|^{6/(3-2\beta)} dx \right)^{(3-2\beta)/(12\beta)} \\
&\leq C C_0^{(2\beta-1)/(4\beta)} \|u_0\|_{\dot{H}^\beta}^{1/(2\beta)} \leq C C_0^{(2\beta-1)/(4\beta)} \leq C C_0^{3\delta_0/2}.
\end{aligned} \tag{3.87}$$

since it follows from (3.8) that $3\delta_0/2 \leq (2\beta-1)/(4\beta)$. So, if C_0 is chosen to be such that

$$C_0 \leq \varepsilon_6 \triangleq \min \left\{ \varepsilon_5, (2C_1)^{-2/\delta_0} \right\},$$

then (3.82) immediately follows from (3.86). \square

We can now close the estimates of $A_1(T)$ and $A_2(T)$.

Lemma 3.7 *Let (ρ, u, d) be a smooth solution of (1.1)–(1.7) on $\mathbb{R}^3 \times (0, T]$ satisfying (3.7). Then there exists a constant $\varepsilon_7 > 0$, depending only on $\mu, \lambda, A, \gamma, \bar{\rho}, \tilde{\rho}, \beta, M_1$ and M_2 , such that*

$$A_1(T) + A_2(T) \leq C_0^{1/2}, \tag{3.88}$$

provided $C_0 \leq \varepsilon_7$.

Proof. To prove (3.88), we first utilize Lemmas 2.1, 2.2, (3.11) and (3.15) to get that

$$\begin{aligned}
\|F\|_{L^4}^4 + \|\omega\|_{L^4}^4 &\leq C \left(\|F\|_{L^2} \|\nabla F\|_{L^2}^3 + \|\omega\|_{L^2} \|\nabla \omega\|_{L^2}^3 \right) \\
&\leq C \left(\|\nabla u\|_{L^2} + C_0^{1/2} \right) \left(\|\rho^{1/2} \dot{u}\|_{L^2}^3 + C_0^{\delta_0} \|\nabla^3 d\|_{L^2}^3 \right),
\end{aligned}$$

from which it follows that

$$\begin{aligned}
& \int_0^T \sigma^2 (\|F\|_{L^4}^4 + \|\omega\|_{L^4}^4) dt \\
& \leq C \int_0^T \sigma^2 \|\nabla u\|_{L^2} \|\rho^{1/2} \dot{u}\|_{L^2}^3 dt + CC_0^{\delta_0} \int_0^T \sigma^2 \|\nabla u\|_{L^2} \|\nabla^3 d\|_{L^2}^3 dt \\
& \quad + CC_0^{1/2} \int_0^T \sigma^2 \|\rho^{1/2} \dot{u}\|_{L^2}^3 dt + CC_0^{1/2+\delta_0} \int_0^T \sigma^2 \|\nabla^3 d\|_{L^2}^3 dt \\
& \triangleq \sum_{i=1}^4 R_i.
\end{aligned} \tag{3.89}$$

In view of the definitions of $A_i(T)$ ($i = 1, \dots, 5$) in (3.1)–(3.5), we deduce from (3.7), (3.53) and Hölder inequality that

$$\begin{aligned}
R_1 & \leq C \int_0^{\sigma(T)} \sigma^{(6\beta-7)/8} \left(\sigma^{(3-2\beta)/4} \|\nabla u\|_{L^2}^2 \right)^{1/2} \left(\sigma^2 \|\rho^{1/2} \dot{u}\|_{L^2}^2 \right) \left(\sigma^{1-\beta} \|\rho^{1/2} \dot{u}\|_{L^2}^2 \right)^{1/2} dt \\
& \quad + C \int_{\sigma(T)}^T (\sigma \|\nabla u\|_{L^2}^2)^{1/2} \left(\sigma^2 \|\rho^{1/2} \dot{u}\|_{L^2}^2 \right)^{1/2} \left(\sigma \|\rho^{1/2} \dot{u}\|_{L^2}^2 \right) dt \\
& \leq CA_2(\sigma(T)) A_4^{1/2}(\sigma(T)) \left(\int_0^{\sigma(T)} \sigma^{1-\beta} \|\rho^{1/2} \dot{u}\|_{L^2}^2 dt \right)^{1/2} \left(\int_0^{\sigma(T)} \sigma^{(6\beta-7)/4} dt \right)^{1/2} \\
& \quad + CA_1^{1/2}(T) A_2^{1/2}(T) \int_{\sigma(T)}^T \sigma \|\rho^{1/2} \dot{u}\|_{L^2}^2 dt \\
& \leq CC_0^{(1+\delta_0)/2} + CC_0 \leq CC_0^{(1+\delta_0)/2}.
\end{aligned} \tag{3.90}$$

Similarly, due to (3.7) and (3.21), we have

$$\begin{aligned}
\sum_{i=2}^4 R_i & \leq CC_0^{\delta_0} \int_0^T (\sigma \|\nabla u\|_{L^2}^2)^{1/2} (\sigma^2 \|\nabla^3 d\|_{L^2}^2)^{1/2} \left(\sigma^{1-\beta} \|\nabla^3 d\|_{L^2}^2 \right) dt \\
& \quad + CC_0^{1/2} \int_0^T \left(\sigma^2 \|\rho^{1/2} \dot{u}\|_{L^2}^2 \right)^{1/2} \left(\sigma \|\rho^{1/2} \dot{u}\|_{L^2}^2 \right) dt \\
& \quad + CC_0^{1/2+\delta_0} \int_0^T (\sigma^2 \|\nabla^3 d\|_{L^2}^2)^{1/2} (\sigma \|\nabla^3 d\|_{L^2}^2) dt \\
& \leq C \left(C_0^{\delta_0} A_1^{1/2}(T) A_2^{1/2}(T) + C_0^{1/2} A_1(T) A_2^{1/2}(T) + C_0^{1/2+\delta_0} A_1(T) A_2^{1/2}(T) \right) \\
& \leq CC_0^{1/2+\delta_0}.
\end{aligned} \tag{3.91}$$

Thus, substituting (3.90), (3.91) into (3.89) gives

$$\int_0^T \sigma^2 (\|F\|_{L^4}^4 + \|\omega\|_{L^4}^4) dt \leq CC_0^{(1+\delta_0)/2}. \tag{3.92}$$

In terms of the effective viscous flux F , we can rewrite (3.35) as

$$\begin{aligned}
0 & = (P(\rho) - P(\tilde{\rho}))_t + u \cdot \nabla (P(\rho) - P(\tilde{\rho})) + \gamma P(\tilde{\rho}) \operatorname{div} u + \gamma (P(\rho) - P(\tilde{\rho})) \operatorname{div} u \\
& = (P(\rho) - P(\tilde{\rho}))_t + u \cdot \nabla (P(\rho) - P(\tilde{\rho})) + \gamma P(\tilde{\rho}) \operatorname{div} u
\end{aligned}$$

$$+\frac{\gamma}{2\mu+\lambda}(P(\rho)-P(\tilde{\rho}))F+\frac{\gamma}{2\mu+\lambda}(P(\rho)-P(\tilde{\rho}))^2, \quad (3.93)$$

which, multiplied by $3\sigma^2(P(\rho)-P(\tilde{\rho}))^2$ and integrated by parts over $\mathbb{R}^3 \times (0, T)$, yields

$$\begin{aligned} & \frac{3\gamma-1}{2\mu+\lambda} \int_0^T \sigma^2 \|P(\rho)-P(\tilde{\rho})\|_{L^4}^4 dt \\ & \leq C\sigma^2 \|P(\rho)-P(\tilde{\rho})\|_{L^3}^3 + C \int_0^T (\sigma' \|P(\rho)-P(\tilde{\rho})\|_{L^3}^3 + \sigma^2 \|\nabla u\|_{L^2}^2 + \sigma^2 \|F\|_{L^4}^4) dt \\ & \leq CC_0^{(1+\delta_0)/2}. \end{aligned} \quad (3.94)$$

where we have used (3.11), (3.92) and Cauchy-Schwarz inequality.

As a result of (3.92) and (3.94), we deduce from the standard L^p -estimate that

$$\begin{aligned} \int_0^T \sigma^2 \|\nabla u\|_{L^4}^4 dt & \leq C \int_0^T \sigma^2 (\|\operatorname{div} u\|_{L^4}^4 + \|\omega\|_{L^4}^4) dt \\ & \leq C \int_0^T \sigma^2 (\|F\|_{L^4}^4 + \|P(\rho)-P(\tilde{\rho})\|_{L^4}^4 + \|\omega\|_{L^4}^4) dt \\ & \leq CC_0^{(1+\delta_0)/2}. \end{aligned} \quad (3.95)$$

Thus, it follows from (3.31), (3.32), (3.94) and (3.95) that

$$A_1(T) + A_2(T) \leq CC_0^{1/2+2\delta_0} + CC_0^{(1+\delta_0)/2} \leq C_1 C_0^{(1+\delta_0)/2} \leq C_0^{1/2},$$

provided C_0 is chosen to be such that

$$C_0 \leq \varepsilon_7 \triangleq \min \left\{ \varepsilon_6, C_1^{-2/\delta_0} \right\}.$$

The proof of Lemma 3.7 is therefore completed. \square

Finally, we need to prove the uniform upper bound of the density.

Lemma 3.8 *Let (ρ, u, d) be a smooth solution of (1.1)–(1.7) on $\mathbb{R}^3 \times (0, T]$ satisfying (3.7). Then there exists a constant $\varepsilon_8 > 0$, depending only on $\mu, \lambda, A, \gamma, \bar{\rho}, \tilde{\rho}, \beta, M_1$ and M_2 , such that*

$$\sup_{0 \leq t \leq T} \|\rho\|_{L^\infty} \leq \frac{7}{4} \bar{\rho} \quad (3.96)$$

provided $C_0 \leq \varepsilon_8$.

Proof. Let $D_t \triangleq \partial_t + u \cdot \nabla$ denote the material derivative operator. Then, in terms of the effective viscous flux F in (2.3), we can rewrite (1.1) as

$$D_t \rho = g(\rho) + b'(\rho)$$

where

$$g(\rho) = -\frac{A\rho}{2\mu+\lambda}(\rho^\gamma - \tilde{\rho}^\gamma), \quad b(t) = -\frac{1}{2\mu+\lambda} \int_0^t \rho F ds.$$

Thus, to apply Lemma 2.3, we now need to estimate $b(t)$. To do this, we first use (2.2) and (2.7) to deduce that for any $0 \leq t_1 < t_2 \leq \sigma(T)$,

$$\begin{aligned}
|b(t_2) - b(t_1)| &\leq C \int_0^{\sigma(T)} \|F\|_{L^\infty} dt \\
&= C \int_0^{\sigma(T)} \|\Delta^{-1} \operatorname{div}(\rho \dot{u}) + \Delta^{-1} \operatorname{div} \operatorname{div}(M(d))\|_{L^\infty} dt \\
&\leq C \int_0^{\sigma(T)} \left(\|\rho \dot{u}\|_{L^2}^{1/2} \|\dot{u}\|_{L^6}^{1/2} + \|M(d)\|_{L^{3/2}}^{1/5} \|\nabla M(d)\|_{L^6}^{4/5} \right) dt \\
&\leq C \int_0^{\sigma(T)} \|\rho \dot{u}\|_{L^2}^{1/2} \|\nabla \dot{u}\|_{L^2}^{1/2} dt + C \int_0^{\sigma(T)} \|\nabla d\|_{L^3}^{2/3} \|\nabla^2 d\|_{L^6}^{4/3} dt \\
&\triangleq R_1 + R_2.
\end{aligned} \tag{3.97}$$

On one hand, using (3.53), (3.54) and (3.88), we find

$$\begin{aligned}
R_1 &\leq C \int_0^{\sigma(T)} t^{(\beta-2)/4} \|\rho^{1/2} \dot{u}\|_{L^2}^{1/2} \left(t^{2-\beta} \|\nabla \dot{u}\|_{L^2}^2 \right)^{1/4} dt \\
&\leq C \left(\int_0^{\sigma(T)} t^{(\beta-2)/3} \|\rho^{1/2} \dot{u}\|_{L^2}^{2/3} dt \right)^{3/4} \left(\int_0^{\sigma(T)} t^{2-\beta} \|\nabla \dot{u}\|_{L^2}^2 dt \right)^{1/4} \\
&\leq C \left(\int_0^{\sigma(T)} t^{-[(2-\beta)(2/3-\delta_0)+\delta_0]} \left(t^{2-\beta} \|\rho^{1/2} \dot{u}\|_{L^2}^2 \right)^{1/3-\delta_0} \left(t \|\rho^{1/2} \dot{u}\|_{L^2}^2 \right)^{\delta_0} dt \right)^{3/4} \\
&\leq C \left(\int_0^{\sigma(T)} t^{-[2(2-\beta)+3(\beta-1)\delta_0]/(3(1-\delta_0))} dt \right)^{3(1-\delta_0)/4} \left(\int_0^{\sigma(T)} t \|\rho^{1/2} \dot{u}\|_{L^2}^2 dt \right)^{3\delta_0/4} \\
&\leq C A_1^{3\delta_0/4}(T) \leq C C_0^{3\delta_0/8},
\end{aligned}$$

since it follows from (3.8) that

$$-\frac{2(2-\beta) + 3(\beta-1)\delta_0}{3(1-\delta_0)} = -\frac{9\beta - 4\beta^2 + 1}{7\beta + 1} > -1.$$

On the other hand, it follows from (3.15) and (3.21) that

$$\begin{aligned}
R_2 &\leq C C_0^{2\delta_0/3} \int_0^{\sigma(T)} t^{-2(1-\beta)/3} \left(t^{1-\beta} \|\nabla^3 d\|_{L^2}^2 \right)^{2/3} dt \\
&\leq C C_0^{2\delta_0/3} \left(\int_0^{\sigma(T)} t^{2(\beta-1)} dt \right)^{1/3} \left(\int_0^{\sigma(T)} t^{1-\beta} \|\nabla^3 d\|_{L^2}^2 dt \right)^{2/3} \\
&\leq C C_0^{2\delta_0/3}.
\end{aligned}$$

Thus, putting the estimates of R_1, R_2 into (3.97) gives

$$|b(t_2) - b(t_1)| \leq C C_0^{\delta_0/3}.$$

So, for $t \in [0, \sigma(T)]$ one can choose N_0, N_1 and ξ^* in Lemma 2.3 as follows:

$$N_0 = C C_0^{\delta_0/3}, \quad N_1 = 0, \quad \xi^* = \tilde{\rho}.$$

Since it holds that

$$g(\xi) = -\frac{A\xi}{2\mu + \lambda} (\xi^\gamma - \tilde{\rho}^\gamma) \leq -N_1 = 0, \quad \forall \xi \geq \xi^* = \tilde{\rho},$$

we thus conclude from (2.9) that

$$\sup_{0 \leq t \leq \sigma(T)} \|\rho\|_{L^\infty} \leq \max\{\bar{\rho}, \tilde{\rho}\} + N_0 \leq \bar{\rho} + CC_0^{\delta_0/3} \leq \frac{3}{2}\bar{\rho}, \quad (3.98)$$

provided C_0 is chosen to be such that

$$C_0 \leq \varepsilon_{8,1} \triangleq \min \left\{ \varepsilon_7, \left(\frac{\bar{\rho}}{2C} \right)^{3/\delta_0} \right\}.$$

On the other hand, it follows from Lemmas 2.1, 2.2, (3.7), (3.11) and (3.88) that

$$\begin{aligned} \|F\|_{L^\infty} &\leq C\|F\|_{L^2}^{1/4} \|\nabla F\|_{L^6}^{3/4} \leq CC_0^{1/16} \|\nabla F\|_{L^6}^{3/4} \\ &\leq CC_0^{1/16} \left(\|\nabla \dot{u}\|_{L^2} + \|\nabla^2 d\|_{L^2}^{1/2} \|\nabla^3 d\|_{L^2}^{3/2} \right)^{3/4} \end{aligned}$$

for any $t \in [\sigma(T), T]$. Hence, using (3.22) and (3.88), we deduce that for any $t_1, t_2 \in [\sigma(T), T]$,

$$\begin{aligned} |b(t_2) - b(t_1)| &\leq C \int_{t_1}^{t_2} \|F\|_{L^\infty} dt \leq \frac{A}{2\mu + \lambda} (t_2 - t_1) + C \int_{\sigma(T)}^T \|F\|_{L^\infty}^{8/3} dt \\ &\leq \frac{A}{2\mu + \lambda} (t_2 - t_1) + CC_0^{1/6} \int_{\sigma(T)}^T \left(\sigma^2 \|\nabla \dot{u}\|_{L^2}^2 + \sigma^{5/2} \|\nabla^2 d\|_{L^2} \|\nabla^3 d\|_{L^2}^3 \right) dt \\ &\leq \frac{A}{2\mu + \lambda} (t_2 - t_1) + CC_0^{1/6}. \end{aligned}$$

Thus, for $t \in [\sigma(T), T]$ one can choose N_0, N_1 and ξ^* in Lemma 2.3 as follows:

$$N_0 = CC_0^{1/6}, \quad N_1 = \frac{A}{2\mu + \lambda}, \quad \xi^* = \tilde{\rho} + 1.$$

Noting that

$$g(\xi) = -\frac{A\xi}{2\mu + \lambda} (\xi^\gamma - \tilde{\rho}^\gamma) \leq -N_1 = -\frac{A}{2\mu + \lambda}, \quad \forall \xi \geq \xi = \tilde{\rho} + 1,$$

we can thus apply Lemma 2.3 to get

$$\sup_{\sigma(T) \leq t \leq T} \|\rho\|_{L^\infty} \leq \max \left\{ \frac{3\bar{\rho}}{2}, \tilde{\rho} + 1 \right\} + N_0 \leq \frac{3\bar{\rho}}{2} + CC_0^{1/6} \leq \frac{7}{4}\bar{\rho}, \quad (3.99)$$

provided C_0 is chosen to be such that

$$C_0 \leq \varepsilon_8 \triangleq \min \left\{ \varepsilon_{8,1}, \left(\frac{\bar{\rho}}{4C} \right)^6 \right\}.$$

Therefore, the combination of (3.98) with (3.99) ends the proof of Lemma 3.8. \square

Now, by virtue of Lemmas 3.1–3.8 we can complete the proof of Proposition 3.1.

Proof of Proposition 3.1. By Lemmas 3.2, 3.6-3.8, to prove Proposition 3.1, it remains to estimate the term $A_4(\sigma(T))$. In fact, using (3.21), (3.53) and (3.88), we have

$$\begin{aligned}
A_4(\sigma(T)) &\leq \sup_{0 \leq t \leq \sigma(T)} \left(\sigma^{1-\beta} \|\nabla u\|_{L^2}^2 \right)^{(2\beta+1)/(4\beta)} \sup_{0 \leq t \leq \sigma(T)} \left(\sigma \|\nabla u\|_{L^2}^2 \right)^{(2\beta-1)/(4\beta)} \\
&\quad + \sup_{0 \leq t \leq \sigma(T)} \left(\sigma^{1-\beta} \|\nabla^2 d\|_{L^2}^2 \right)^{(2\beta+1)/(4\beta)} \sup_{0 \leq t \leq \sigma(T)} \left(\sigma \|\nabla^2 d\|_{L^2}^2 \right)^{(2\beta-1)/(4\beta)} \\
&\quad + \left(\int_0^{\sigma(T)} \sigma^{1-\beta} \|\rho^{1/2} \dot{u}\|_{L^2}^2 dt \right)^{(2\beta+1)/(4\beta)} \left(\int_0^{\sigma(T)} \sigma \|\rho^{1/2} \dot{u}\|_{L^2}^2 dt \right)^{(2\beta-1)/(4\beta)} \\
&\quad + \left(\int_0^{\sigma(T)} \sigma^{1-\beta} \|\nabla^3 d\|_{L^2}^2 dt \right)^{(2\beta+1)/(4\beta)} \left(\int_0^{\sigma(T)} \sigma \|\nabla^3 d\|_{L^2}^2 dt \right)^{(2\beta-1)/(4\beta)} \\
&\leq C A_1^{(2\beta-1)/(4\beta)}(T) \leq C C_0^{(2\beta-1)/(8\beta)} \leq \frac{C_0^{\delta_0}}{2},
\end{aligned}$$

provided C_0 is chosen to be such that

$$C_0 \leq \varepsilon \triangleq \min \left\{ \varepsilon_8, (2C)^{-72\beta/(2\beta-1)} \right\}.$$

Therefore, the proof of Proposition 3.1 is completed. \square

4 Time-dependent higher-order estimates

In this section, we prove the global estimates on the spatial-time derivatives of (ρ, u, d) which are needed to guarantee the existence of classical solutions. For this purpose, we assume that the conditions of Theorem 1.1 hold and that (ρ, u, d) is a smooth solution of (1.1)–(1.7) on $\mathbb{R}^3 \times (0, T]$ satisfying Proposition 3.1. Moreover, from now on we will always assume that the initial energy C_0 satisfies (3.10). For simplicity, throughout this section we denote by C or C_i ($i = 1, 2, 3, \dots$) the various positive constants which may depend on

$$\mu, \lambda, A, \gamma, \bar{\rho}, \tilde{\rho}, \beta, M_1, M_2, g \text{ and } T,$$

where $g \in L^2$ is the function in the compatibility condition (1.9) and $T > 0$ is the time.

First, one easily deduces from Lemmas 3.3–3.5 that

Lemma 4.1 *Assume that the conditions of Theorem 1.1 hold. Then for any given $T > 0$, there exists a constant $C(T) > 0$ such that*

$$\begin{aligned}
&\sup_{0 \leq t \leq T} (\|\nabla^2 d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2) \\
&\quad + \int_0^T \left(\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 \right) dt \leq C(T)
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
&\sup_{0 \leq t \leq T} \left(\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 \right) \\
&\quad + \int_0^T (\|\nabla \dot{u}\|_{L^2}^2 + \|d_{tt}\|_{L^2}^2 + \|\nabla^2 d_t\|_{L^2}^2) dt \leq C(T).
\end{aligned} \tag{4.2}$$

Proof. Indeed, choosing $\beta = 1$ in (3.21) and (3.53), one immediately obtains (4.1) from (3.21), (3.53) and (3.88).

In order to prove (4.2), we first notice that (3.52), together with (4.1), implies

$$\|\nabla^3 d\|_{L^2}^2 \leq C(1 + \|\nabla d_t\|_{L^2}^2).$$

Taking this into account, choosing $m = 0$ in (3.46), (3.50) and integrating them over $(0, T)$, we have by (4.1) and the compatibility condition (1.9) that

$$\begin{aligned} & \sup_{0 \leq t \leq T} \left(\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 \right) \\ & + \int_0^T (\|\nabla \dot{u}\|_{L^2}^2 + \|d_{tt}\|_{L^2}^2 + \|\nabla^2 d_t\|_{L^2}^2) dt \\ & \leq C + C \int_0^T \|\nabla u\|_{L^4}^4 dt \leq C + C \int_0^T \|\nabla u\|_{L^2} \|\nabla u\|_{L^6}^3 dt \\ & \leq C + C \int_0^T \left(\|\rho^{1/2} \dot{u}\|_{L^2}^3 + \|\nabla^3 d\|_{L^2}^3 \right) dt, \end{aligned} \quad (4.3)$$

where we have also used (2.1), (2.6) and (3.15). Thus, in view of (4.1) and (4.3), an application of Gronwall's inequality immediately leads to (4.2). \square

Next, we use the Beale-Kato-Majda-type inequality (see Lemma 2.4) to derive the estimates on the spatial gradient of the density as well as the velocity, which are very important for the proof of the higher-order estimates of the solutions. The proofs are mainly motivated by the ones in [17, 19].

Lemma 4.2 *Assume that the conditions of Theorem 1.1 hold. Then for any given $T > 0$, there exists a constant $C(T) > 0$ such that*

$$\sup_{0 \leq t \leq T} (\|\nabla \rho\|_{L^2 \cap L^6} + \|\nabla u\|_{H^1}) + \int_0^T \|\nabla u\|_{L^\infty}^{3/2} dt \leq C(T). \quad (4.4)$$

Proof. For $2 \leq p \leq 6$, it is easily derived from (1.1) that

$$\begin{aligned} \frac{d}{dt} \|\nabla \rho\|_{L^p} & \leq C \|\nabla u\|_{L^\infty} \|\nabla \rho\|_{L^p} + C \|\nabla^2 u\|_{L^p} \\ & \leq C(1 + \|\nabla u\|_{L^\infty}) \|\nabla \rho\|_{L^p} + C(1 + \|\rho \dot{u}\|_{L^p}), \end{aligned} \quad (4.5)$$

where we have used (4.1), (4.2) and the L^p -estimate of elliptic system to infer from (1.2) that

$$\begin{aligned} \|\nabla^2 u\|_{L^p} & \leq C(\|\rho \dot{u}\|_{L^p} + \|\nabla \rho\|_{L^p} + \|\nabla d \nabla^2 d\|_{L^p}) \\ & \leq C(\|\rho \dot{u}\|_{L^p} + \|\nabla \rho\|_{L^p} + 1). \end{aligned} \quad (4.6)$$

We are now in a position of estimating $\|\nabla u\|_{L^\infty}$. By (2.1), (4.1) and (4.6), we deduce from (2.11) with $q = 6$ that

$$\begin{aligned} \|\nabla u\|_{L^\infty} & \leq C + C(\|\operatorname{div} u\|_{L^\infty} + \|\nabla \times u\|_{L^\infty}) \ln(e + \|\rho \dot{u}\|_{L^6} + \|\nabla \rho\|_{L^6}) \\ & \leq C + C(\|\operatorname{div} u\|_{L^\infty} + \|\nabla \times u\|_{L^\infty}) \ln(e + \|\nabla \dot{u}\|_{L^2}) \\ & \quad + C(\|\operatorname{div} u\|_{L^\infty} + \|\nabla \times u\|_{L^\infty}) \ln(e + \|\nabla \rho\|_{L^6}). \end{aligned} \quad (4.7)$$

So if we set

$$\Phi(t) \triangleq e + \|\nabla \rho\|_{L^6}, \quad \Psi(t) \triangleq 1 + (\|\operatorname{div} u\|_{L^\infty} + \|\nabla \times u\|_{L^\infty}) \ln(e + \|\nabla \dot{u}\|_{L^2}) + \|\nabla \dot{u}\|_{L^2},$$

then it follows from (4.5) with $p = 6$ and (4.7) that

$$\Phi'(t) \leq C\Psi(t)\Phi(t) \ln \Phi(t),$$

due to the fact that $\ln \Phi(T) \geq 1$. This particularly implies

$$\frac{d}{dt} \ln \Phi(t) \leq C\Psi(t) \ln \Phi(t). \quad (4.8)$$

By virtue of Lemmas 2.1, 2.2, (4.1) and (4.2), we find ($\omega = \nabla \times u$)

$$\begin{aligned} \int_0^T \Psi^{3/2}(t) dt &\leq C + C \int_0^T (\|\nabla \dot{u}\|_{L^2}^2 + \|\operatorname{div} u\|_{L^\infty}^2 + \|\omega\|_{L^\infty}^2) dt \\ &\leq C + C \int_0^T (\|P(\rho) - P(\tilde{\rho})\|_{L^\infty}^2 + \|F\|_{L^\infty}^2 + \|\omega\|_{L^\infty}^2) dt \\ &\leq C + C \int_0^T (\|F\|_{L^2}^2 + \|\omega\|_{L^2}^2 + \|\nabla F\|_{L^6}^2 + \|\nabla \omega\|_{L^6}^2) dt \\ &\leq C + C \int_0^T (\|\nabla \dot{u}\|_{L^2}^2 + \|\nabla d\|_{L^\infty}^2 \|\nabla^3 d\|_{L^2}^2) dt \leq C, \end{aligned} \quad (4.9)$$

and consequently, it follows from (4.8) and Gronwall's inequality that

$$\sup_{0 \leq t \leq T} \|\nabla \rho(t)\|_{L^6} \leq \sup_{0 \leq t \leq T} \Phi(t) \leq C. \quad (4.10)$$

As a result of (4.7), (4.9) and (4.10), we obtain

$$\int_0^T \|\nabla u\|_{L^\infty}^{3/2} dt \leq C + C \int_0^T \Psi^{3/2}(t) dt \leq C. \quad (4.11)$$

Using (4.1) and (4.11), we also infer from (4.5) with $p = 2$ and Gronwall's inequality that

$$\sup_{0 \leq t \leq T} \|\nabla \rho(t)\|_{L^2} \leq C,$$

which, together with (4.2) and (4.6), yields $\|\nabla u\|_{H^1} \leq C$. The proof of Lemma 4.2 is therefore completed. \square

Basing on Lemmas 4.1 and 4.2, we easily obtain

Lemma 4.3 *Assume that the conditions of Theorem 1.1 hold. Then for any given $T > 0$,*

$$\sup_{0 \leq t \leq T} \|\rho^{1/2} u_t\|_{L^2}^2 + \int_0^T \|\nabla u_t\|_{L^2}^2 dt \leq C(T), \quad (4.12)$$

$$\sup_{0 \leq t \leq T} (\|\nabla \rho\|_{H^1} + \|\nabla P\|_{H^1}) + \int_0^T \|\nabla^2 u\|_{H^1}^2 dt \leq C(T), \quad (4.13)$$

$$\sup_{0 \leq t \leq T} (\|\rho_t\|_{H^1} + \|P_t\|_{H^1}) + \int_0^T (\|\rho_{tt}\|_{L^2}^2 + \|P_{tt}\|_{L^2}^2) dt \leq C(T). \quad (4.14)$$

Proof. First, due to $u_t = \dot{u} - u \cdot \nabla u$, one easily gets (4.12) from Lemmas 2.1, 4.1 and 4.2.

Next we prove (4.13). To do this, noting that $P(\rho) = A\rho^\gamma$ satisfies

$$P_t + u \cdot \nabla P + \gamma P \operatorname{div} u = 0, \quad (4.15)$$

from which and (1.1) we have by some direct computations that

$$\begin{aligned} & \frac{d}{dt} (\|\nabla^2 \rho\|_{L^2}^2 + \|\nabla^2 P\|_{L^2}^2) \\ & \leq C \|\nabla u\|_{L^\infty} (\|\nabla^2 \rho\|_{L^2}^2 + \|\nabla^2 P\|_{L^2}^2) + C \|\nabla^3 u\|_{L^2} (\|\nabla^2 \rho\|_{L^2} + \|\nabla^2 P\|_{L^2}) \\ & \quad + C \|\nabla^2 u\|_{L^6} (\|\nabla \rho\|_{L^3} + \|\nabla P\|_{L^3}) (\|\nabla^2 \rho\|_{L^2} + \|\nabla^2 P\|_{L^2}) \\ & \leq C (1 + \|\nabla u\|_{L^\infty}) (\|\nabla^2 \rho\|_{L^2}^2 + \|\nabla^2 P\|_{L^2}^2) + C \|\nabla^2 u\|_{H^1}^2, \end{aligned} \quad (4.16)$$

where we have used (2.1) and (4.4). Using (4.1), (4.2), (4.4), (4.12) and Lemma 2.1, we deduce from (1.2) and the standard L^2 -estimate of elliptic system that

$$\begin{aligned} \|\nabla^2 u\|_{H^1} & \leq C (\|\rho \dot{u}\|_{H^1} + \|\nabla P\|_{H^1} + \|\nabla d \nabla^2 d\|_{H^1}) \\ & \leq C (1 + \|\nabla u_t\|_{L^2} + \|\nabla^2 P\|_{L^2}). \end{aligned} \quad (4.17)$$

Thus, combining (4.17) with (4.16) and using Gronwall inequality, we immediately arrive at (4.13) since it follows from (4.4) and (4.12) that

$$\|\nabla u\|_{L^\infty}^{3/2} + \|\nabla u_t\|_{L^2}^2 \in L^1(0, T).$$

Finally, it is easily seen from (1.1) and (4.15) that

$$\begin{aligned} \|\rho_t\|_{H^1} + \|P_t\|_{H^1} & \leq C \|u\|_{L^\infty} (\|\nabla \rho\|_{H^1} + \|\nabla P\|_{H^1}) + C \|\nabla u\|_{H^1} \\ & \quad + C \|\nabla u\|_{L^6} (\|\nabla \rho\|_{L^3} + \|\nabla P\|_{L^3}) \\ & \leq C (1 + \|\nabla u\|_{H^1}^2 + \|\nabla \rho\|_{H^1}^2 + \|\nabla P\|_{H^1}^2) \leq C, \end{aligned} \quad (4.18)$$

where we have used (4.4), (4.13) and Lemma 2.1. Moreover, noticing that (4.15) implies

$$P_{tt} + u_t \cdot \nabla P + u \cdot \nabla P_t + \gamma P_t \operatorname{div} u + \gamma P \operatorname{div} u_t = 0,$$

and hence, using (4.4), (4.12), (4.13) and (4.18), one obtains

$$\begin{aligned} \int_0^T \|P_{tt}\|_{L^2}^2 dt & \leq C \int_0^T (\|u_t\|_{L^6} \|\nabla P\|_{L^3} + \|u\|_{W^{1,\infty}} \|P_t\|_{H^1} + \|\nabla u_t\|_{L^2})^2 dt \\ & \leq C + C \int_0^T (\|\nabla u\|_{H^2}^2 + \|\nabla u_t\|_{L^2}^2) dt \leq C. \end{aligned} \quad (4.19)$$

Analogously, one also has $\|\rho_{tt}\|_{L^2} \in L^2(0, T)$. So, combining this with (4.18), (4.19) completes the proof of (4.14). \square

Due to the weaker compatibility condition (1.9), the methods used in [17] to derive the higher-order estimates on the solutions (ρ, u, d) , which guarantee the solutions obtained are indeed a classical one away from the initial time, cannot be applied any more. To overcome this difficulty, we need some careful initial-layer analysis.

Lemma 4.4 *Let $\sigma \triangleq \min\{1, t\}$. Then for any given $T > 0$,*

$$\begin{aligned} & \sup_{0 \leq t \leq T} \sigma \left(\|\nabla u\|_{H^2}^2 + \|\nabla u_t\|_{L^2}^2 + \|\nabla^2 d_t\|_{L^2}^2 + \|\nabla^2 d\|_{H^2}^2 \right) + \int_0^T \|\nabla^2 d\|_{H^2}^2 dt \\ & + \int_0^T \sigma \left(\|\rho^{1/2} u_{tt}\|_{L^2}^2 + \|\nabla u_t\|_{H^1}^2 + \|\nabla d_{tt}\|_{L^2}^2 \right) dt \leq C(T). \end{aligned} \quad (4.20)$$

Proof. Differentiating (1.2) and (1.3) with respect to t gives

$$\rho u_{tt} - \mu \Delta u_t - (\mu + \lambda) \nabla \operatorname{div} u_t = -\rho_t u_t - (\rho u \cdot \nabla u)_t - \nabla P_t - \operatorname{div} M(d)_t \quad (4.21)$$

and

$$d_{tt} - \Delta d_t = (|\nabla d|^2 d)_t - (u \cdot \nabla d)_t. \quad (4.22)$$

Thus, multiplying (4.21) and (4.22) by u_{tt} and $-\Delta d_{tt}$ respectively, and integrating the resulting equations over \mathbb{R}^3 , we obtain after adding them together that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int (\mu |\nabla u_t|^2 + (\mu + \lambda) (\operatorname{div} u_t)^2 + |\Delta d_t|^2) dx + \int (\rho |u_{tt}|^2 + |\nabla d_{tt}|^2) dx \\ & = - \int (\rho_t u_t + \rho_t u \cdot \nabla u + \rho u_t \cdot \nabla u + \rho u \cdot \nabla u_t + \nabla P_t + \operatorname{div} M(d)_t) \cdot u_{tt} dx \\ & \quad + \int (u_t \cdot \nabla d + u \cdot \nabla d_t - |\nabla d|^2 d_t - 2d(\nabla d) \cdot (\nabla d_t)) \cdot \Delta d_{tt} dx \\ & = - \frac{d}{dt} \int \left[\frac{1}{2} \rho_t |u_t|^2 + (\rho_t u \cdot \nabla u + \nabla P_t + \operatorname{div} M(d)_t) \cdot u_t \right] dx \\ & \quad + \int (\rho_{tt} u \cdot \nabla u + \rho_t u_t \cdot \nabla u + \rho_t u \cdot \nabla u_t + \operatorname{div} M(d)_{tt}) \cdot u_t dx \\ & \quad + \frac{1}{2} \int \rho_{tt} |u_t|^2 dx - \int P_{tt} \operatorname{div} u_t dx - \int (\rho u_t \cdot \nabla u + \rho u \cdot \nabla u_t) \cdot u_{tt} dx \\ & \quad + \int [(u_t \cdot \nabla d + u \cdot \nabla d_t - |\nabla d|^2 d_t - 2d(\nabla d) \cdot (\nabla d_t)) \cdot \Delta d_{tt}] dx \\ & \triangleq \frac{d}{dt} I_0 + \sum_{i=1}^5 I_i. \end{aligned} \quad (4.23)$$

We now estimate each term on the right-hand side of (4.23). Using (1.1) and integrating by parts, we first deduce from Lemmas 2.1, 4.1–4.3 and Cauchy-Schwarz inequality that

$$\begin{aligned} I_0 & = - \int [\rho u \cdot (u_t \cdot \nabla u_t) + \rho_t u \cdot \nabla u \cdot u_t - P_t \operatorname{div} u_t - M(d)_t : \nabla u_t] dx \\ & \leq C \|u\|_{L^\infty} \|\rho^{1/2} u_t\|_{L^2} \|\nabla u_t\|_{L^2} + C \|\rho_t\|_{L^2} \|u\|_{L^\infty} \|\nabla u\|_{L^3} \|u_t\|_{L^6} \\ & \quad + C \|P_t\|_{L^2} \|\nabla u_t\|_{L^2} + C \|\nabla d\|_{L^\infty} \|\nabla d_t\|_{L^2} \|\nabla u_t\|_{L^2} \\ & \leq \frac{\mu}{4} \|\nabla u_t\|_{L^2}^2 + C. \end{aligned}$$

Since it holds that $M(d)_{tt} \leq C(|\nabla d_{tt}| |\nabla d| + |\nabla d_t|^2)$, we have by Lemmas 2.1, 4.1–4.3 and integration by parts that

$$I_1 \leq C \|\rho_{tt}\|_{L^2} \|u\|_{L^\infty} \|\nabla u\|_{L^3} \|u_t\|_{L^6} + C \|\rho_t\|_{L^2} \|u_t\|_{L^6}^2 \|\nabla u\|_{L^6}$$

$$\begin{aligned}
& +C\|\rho_t\|_{L^6}\|u\|_{L^6}\|\nabla u_t\|_{L^2}\|u_t\|_{L^6} + C\|\nabla d_{tt}\|_{L^2}\|\nabla d\|_{L^\infty}\|\nabla u_t\|_{L^2} \\
& +C\|\nabla d_t\|_{L^2}^{1/2}\|\nabla d_t\|_{L^6}^{3/2}\|\nabla u_t\|_{L^2} \\
& \leq \frac{1}{4}\|\nabla d_{tt}\|_{L^2}^2 + C\left(\|\rho_{tt}\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 + \|\nabla^2 d_t\|_{L^2}^3\right).
\end{aligned}$$

Due to (1.1), one has $\rho_{tt} = -\operatorname{div}(\rho u)_t$, and hence, it follows from (4.12), (4.14) and integration by parts that

$$\begin{aligned}
I_2 + I_3 &= \int (\rho_t u \cdot \nabla u_t \cdot u_t + \rho u_t \cdot \nabla u_t \cdot u_t) dx - \int P_{tt} \operatorname{div} u_t dx \\
&\leq C(\|\rho_t\|_{L^3}\|u\|_{L^\infty}\|u_t\|_{L^6} + \|\rho u_t\|_{L^3}\|u_t\|_{L^6} + \|P_{tt}\|_{L^2})\|\nabla u_t\|_{L^2} \\
&\leq C\left(\|\nabla u_t\|_{L^2}^2 + \|\rho^{1/2} u_t\|_{L^2}^{1/2}\|u_t\|_{L^6}^{1/2}\|\nabla u_t\|_{L^2}^2 + \|P_{tt}\|_{L^2}^2\right) \\
&\leq C(1 + \|\nabla u_t\|_{L^2}^4 + \|P_{tt}\|_{L^2}^2).
\end{aligned}$$

and similarly, by (4.4) one gets

$$\begin{aligned}
I_4 &\leq C(\|u_t\|_{L^6}\|\nabla u\|_{L^3} + \|u\|_{L^\infty}\|\nabla u_t\|_{L^2})\|\rho^{1/2} u_{tt}\|_{L^2} \\
&\leq \frac{1}{2}\|\rho^{1/2} u_{tt}\|_{L^2}^2 + C\|\nabla u_t\|_{L^2}^2.
\end{aligned}$$

Finally, integrating by parts and using Lemmas 2.1 and 4.1–4.3, we can estimate the last term on the right-hand side of (4.23) as follows:

$$\begin{aligned}
I_5 &\leq C \int (|\nabla u_t| |\nabla d| + |u_t| |\nabla^2 d| + |\nabla u| |\nabla d_t| + |u| |\nabla^2 d_t|) |\nabla d_{tt}| dx \\
&\quad + C \int (|\nabla d| |\nabla^2 d| |d_t| + |\nabla d|^2 |\nabla d_t| + |\nabla^2 d| |\nabla d_t| + |\nabla d| |\nabla^2 d_t|) |\nabla d_{tt}| dx \\
&\leq C\|\nabla d\|_{L^\infty}\|\nabla u_t\|_{L^2}\|\nabla d_{tt}\|_{L^2} + C\|u_t\|_{L^6}\|\nabla^2 d\|_{L^3}\|\nabla d_{tt}\|_{L^2} \\
&\quad + C\|\nabla u\|_{L^3}\|\nabla d_t\|_{L^6}\|\nabla d_{tt}\|_{L^2} + C\|u\|_{L^\infty}\|\nabla^2 d_t\|_{L^2}\|\nabla d_{tt}\|_{L^2} \\
&\quad + C\|\nabla d\|_{L^\infty}\|\nabla^2 d\|_{L^3}\|d_t\|_{L^6}\|\nabla d_{tt}\|_{L^2} + C\|\nabla d\|_{L^\infty}^2\|\nabla d_t\|_{L^2}\|\nabla d_{tt}\|_{L^2} \\
&\quad + C\|\nabla^2 d\|_{L^3}\|\nabla d_t\|_{L^6}\|\nabla d_{tt}\|_{L^2} + C\|\nabla d\|_{L^\infty}\|\nabla^2 d_t\|_{L^2}\|\nabla d_{tt}\|_{L^2} \\
&\leq \frac{1}{4}\|\nabla d_{tt}\|_{L^2}^2 + C(1 + \|\nabla u_t\|_{L^2}^2 + \|\nabla^2 d_t\|_{L^2}^2).
\end{aligned}$$

By virtue of the estimates of I_i ($i = 0, 1, \dots, 5$), we deduce after multiplying (4.23) by $\sigma(t)$, integrating it over $[0, T]$ and using Gronwall's inequality that

$$\sup_{0 \leq t \leq T} \sigma \left(\|\nabla u_t\|_{L^2}^2 + \|\nabla^2 d_t\|_{L^2}^2 \right) + \int_0^T \sigma \left(\|\rho^{1/2} u_{tt}\|_{L^2}^2 + \|\nabla d_{tt}\|_{L^2}^2 \right) dt \leq C, \quad (4.24)$$

where we have also used (4.2), (4.12) and (4.14). This, together with (4.13) and (4.17), gives

$$\sup_{0 \leq t \leq T} (\sigma \|\nabla u\|_{H^2}^2) \leq \sup_{0 \leq t \leq T} \sigma (1 + \|\nabla u_t\|_{L^2}^2 + \|\nabla P\|_{H^1}^2) \leq C. \quad (4.25)$$

On the other hand, by Lemmas 4.1–4.3 we obtain by applying the standard L^2 -estimate to (4.21) that

$$\|\nabla^2 u_t\|_{L^2} \leq C\|\mu \Delta u_t + (\mu + \lambda) \nabla \operatorname{div} u_t\|_{L^2}$$

$$\begin{aligned}
&\leq C \|(\rho u_t + \rho u \cdot \nabla u + \nabla P + \operatorname{div} M(d))_t\|_{L^2} \\
&\leq C \|\rho^{1/2} u_{tt}\|_{L^2} + C \|\rho_t\|_{L^3} \|u_t\|_{L^6} + C \|\rho_t\|_{L^3} \|u\|_{L^\infty} \|\nabla u\|_{L^6} \\
&\quad + C \|u_t\|_{L^6} \|\nabla u\|_{L^3} + C \|u\|_{L^\infty} \|\nabla u_t\|_{L^2} + C \|\nabla P_t\|_{L^2} \\
&\quad + C \|\nabla d\|_{L^\infty} \|\nabla^2 d_t\|_{L^2} + C \|\nabla d_t\|_{L^6} \|\nabla^2 d\|_{L^3} \\
&\leq C \left(1 + \|\nabla u_t\|_{L^2} + \|\rho^{1/2} u_{tt}\|_{L^2} + \|\nabla^2 d_t\|_{L^2}\right), \tag{4.26}
\end{aligned}$$

which, together with (4.24), yields

$$\int_0^T \sigma \|\nabla^2 u_t\|_{L^2}^2 dt \leq C. \tag{4.27}$$

In a similar manner as the derivation of (4.25), we also infer from (1.3) that

$$\begin{aligned}
\|\nabla^4 d\|_{L^2} &\leq C (\|\nabla^2 d_t\|_{L^2} + \|\nabla^2(u \cdot \nabla d)\|_{L^2} + \|\nabla^2(|\nabla d|^2 d)\|_{L^2}) \\
&\leq C (1 + \|\nabla^2 d_t\|_{L^2} + \|\nabla u\|_{H^1} \|\nabla d\|_{H^2} + \|\nabla d\|_{H^2}^4) \\
&\leq C + C \|\nabla^2 d_t\|_{L^2},
\end{aligned}$$

from which, (4.24), and (4.2), it follows that

$$\sup_{0 \leq t \leq T} (\sigma \|\nabla^2 d\|_{H^2}^2) + \int_0^T \|\nabla^2 d\|_{H^2}^2 dt \leq C. \tag{4.28}$$

Therefore, combining (4.24), (4.25), (4.27) and (4.28) finishes the proof of (4.20). \square

To prove the Hölder continuity of the first-order derivatives of density and pressure, we need the following lemma which is concerned with the $W^{1,q}$ -estimate ($q \in (3, 6)$) on the gradients of density and pressure.

Lemma 4.5 *For fixed $q \in (3, 6)$, it holds for any $T > 0$ that*

$$\sup_{0 \leq t \leq T} (\|\nabla \rho\|_{W^{1,q}} + \|\nabla P\|_{W^{1,q}}) + \int_0^T (\|\nabla u_t\|_{L^q}^{p_0} + \|\nabla^2 u\|_{W^{1,q}}^{p_0}) dt \leq C(T), \tag{4.29}$$

where

$$1 \leq p_0 < \frac{4q}{5q-6} \in (1, 2). \tag{4.30}$$

Proof. Operating ∇^2 to both sides of (1.1), (4.15), and multiplying them by $q|\nabla^2 \rho|^{q-2} \nabla^2 \rho$, $q|\nabla^2 P(\rho)|^{q-2} \nabla^2 P(\rho)$, respectively, we obtain after integrating by parts over \mathbb{R}^3 and using (4.4), (4.13) and Lemma 2.1 that for any $q \in (3, 6)$,

$$\begin{aligned}
&\frac{d}{dt} (\|\nabla^2 \rho\|_{L^q}^q + \|\nabla^2 P\|_{L^q}^q) \\
&\leq C \|\nabla u\|_{L^\infty} (\|\nabla^2 \rho\|_{L^q}^q + \|\nabla^2 P\|_{L^q}^q) + C \|\nabla^2 u\|_{W^{1,q}} \left(\|\nabla^2 \rho\|_{L^q}^{q-1} + \|\nabla^2 P\|_{L^q}^{q-1} \right) \\
&\quad + C \|\nabla^2 u\|_{L^q} \left(\|\nabla \rho\|_{L^\infty} \|\nabla^2 \rho\|_{L^q}^{q-1} + \|\nabla P\|_{L^\infty} \|\nabla^2 P\|_{L^q}^{q-1} \right) \\
&\leq C (1 + \|\nabla u\|_{H^2}) (1 + \|\nabla^2 \rho\|_{L^q}^q + \|\nabla^2 P\|_{L^q}^q) \\
&\quad + C \|\nabla^2 u\|_{W^{1,q}} \left(\|\nabla^2 \rho\|_{L^q}^{q-1} + \|\nabla^2 P\|_{L^q}^{q-1} \right). \tag{4.31}
\end{aligned}$$

Applying the standard L^p -estimate to the elliptic system (1.2) yields that

$$\begin{aligned}
\|\nabla^2 u\|_{W^{1,q}} &\leq C\|\nabla^2 u\|_{L^q} + C\|\nabla(\rho u_t + \rho u \cdot \nabla u + \nabla P + \operatorname{div} M(d))\|_{L^q} \\
&\leq C\|\nabla u\|_{H^2} + C(\|\nabla \rho\|_{L^q}\|u_t\|_{L^\infty} + \|\nabla u_t\|_{L^q}) \\
&\quad + C(\|u\|_{L^\infty}\|\nabla^2 u\|_{L^q} + \|\nabla u\|_{L^\infty}\|\nabla u\|_{L^q} + \|\nabla \rho\|_{L^q}\|u\|_{L^\infty}\|\nabla u\|_{L^\infty}) \\
&\quad + C\|\nabla^2 P\|_{L^q} + C(\|\nabla d\|_{L^\infty}\|\nabla^3 d\|_{L^q} + \|\nabla^2 d\|_{L^\infty}\|\nabla^2 d\|_{L^q}) \\
&\leq C(1 + \|\nabla u_t\|_{L^2} + \|\nabla u_t\|_{L^q} + \|\nabla u\|_{H^2} + \|\nabla^2 d\|_{H^2} + \|\nabla^2 P\|_{L^q}), \tag{4.32}
\end{aligned}$$

where we have also used Lemmas 2.1 and 4.1–4.3. Putting (4.32) into (4.31) gives

$$\begin{aligned}
\frac{d}{dt}(\|\nabla^2 \rho\|_{L^q}^q + \|\nabla^2 P\|_{L^q}^q) &\leq C(1 + \|\nabla u_t\|_{L^2} + \|\nabla u_t\|_{L^q} + \|\nabla u\|_{H^2} + \|\nabla^2 d\|_{H^2}) \\
&\quad \times (1 + \|\nabla^2 \rho\|_{L^q}^q + \|\nabla^2 P\|_{L^q}^q). \tag{4.33}
\end{aligned}$$

For p_0 being the one in (4.30), it follows from (4.12) and (4.20) that

$$\begin{aligned}
&\int_0^T (\|\nabla u_t\|_{L^2} + \|\nabla u\|_{H^2} + \|\nabla^2 d\|_{H^2})^{p_0} dt \\
&\leq C + C \sup_{0 \leq t \leq T} (\sigma \|\nabla u\|_{H^2}^2 + \sigma \|\nabla^2 d\|_{H^2}^2)^{p_0/2} \int_0^T \sigma^{-p_0/2} dt \leq C, \tag{4.34}
\end{aligned}$$

and moreover, using Lemma 2.1, Hölder inequality and (4.20), we find that

$$\begin{aligned}
\int_0^T \|\nabla u_t\|_{L^q}^{p_0} dt &\leq C \int_0^T \|\nabla u_t\|_{L^2}^{p_0(6-q)/(2q)} \|\nabla u_t\|_{L^6}^{p_0(3q-6)/(2q)} dt \\
&\leq C \int_0^T \sigma^{-p_0/2} (\sigma \|\nabla u_t\|_{L^2}^2)^{p_0(6-q)/(4q)} (\sigma \|\nabla u_t\|_{H^1}^2)^{p_0(3q-6)/(4q)} dt \\
&\leq C \left(\sup_{0 \leq t \leq T} \sigma \|\nabla u_t\|_{L^2}^2 \right)^{p_0(6-q)/(4q)} \int_0^T \sigma^{-p_0/2} (\sigma \|\nabla u_t\|_{H^1}^2)^{p_0(3q-6)/(4q)} dt \\
&\leq C \left(\int_0^T \sigma^{-2p_0q/(4q-p_0(3q-6))} dt \right)^{(4q-p_0(3q-6))/(4q)} \left(\int_0^T \sigma \|\nabla u_t\|_{H^1}^2 dt \right)^{p_0(3q-6)/(4q)} \\
&\leq C, \tag{4.35}
\end{aligned}$$

since $0 < 2p_0q/(4q - p_0(3q - 6)) < 1$ and $0 < p_0(3q - 6)/(4q) < 1$.

By virtue of (4.34) and (4.35), we deduce from (4.33) and Gronwall inequality that

$$\sup_{0 \leq t \leq T} (\|\nabla \rho\|_{W^{1,q}} + \|\nabla P\|_{W^{1,q}}) \leq C, \quad \forall q \in (3, 6),$$

which, together with (4.32), (4.34) and (4.35), yields the desired estimate of (4.29). \square

Finally, we need the following initial-layer analysis which particularly implies $(u_t, \nabla^2 u)$ are Hölder continuous away from the initial time $t = 0$.

Lemma 4.6 *For any given $T > 0$, it holds that*

$$\sup_{0 \leq t \leq T} \sigma \left(\|\rho^{1/2} u_{tt}\|_{L^2} + \|\nabla^2 u_t\|_{L^2} + \|\nabla^2 u\|_{W^{1,q}} \right) + \int_0^T \sigma^2 \|\nabla u_{tt}\|_{L^2}^2 dt \leq C(T). \tag{4.36}$$

Proof. Differentiating (4.21) with respect to t gives

$$\begin{aligned} & \rho u_{ttt} + \rho u \cdot \nabla u_{tt} - \mu \Delta u_{tt} - (\mu + \lambda) \nabla \operatorname{div} u_{tt} \\ &= 2 \operatorname{div}(\rho u) u_{tt} + \operatorname{div}(\rho u)_t u_t - 2(\rho u)_t \cdot \nabla u_t - (\rho_{tt} u + 2\rho_t u_t) \cdot \nabla u \\ & \quad - \rho u_{tt} \cdot \nabla u - \nabla P_{tt} - \operatorname{div} M(d)_{tt}, \end{aligned}$$

which, multiplied by u_{tt} in L^2 and integrated by parts over \mathbb{R}^3 , yields

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \rho |u_{tt}|^2 dx + \int (\mu |\nabla u_{tt}|^2 + (\mu + \lambda) (\operatorname{div} u_{tt})^2) dx \\ &= -4 \int \rho u \cdot \nabla u_{tt} \cdot u_{tt} dx - \int (\rho u)_t \cdot (\nabla(u_t \cdot u_{tt}) + 2 \nabla u_t \cdot u_{tt}) dx \\ & \quad - \int (\rho_{tt} u + 2\rho_t u_t) \cdot \nabla u \cdot u_{tt} dx - \int \rho u_{tt} \cdot \nabla u \cdot u_{tt} dx \\ & \quad + \int P_{tt} \operatorname{div} u_{tt} dx + \int M(d)_{tt} : \nabla u_{tt} dx \triangleq \sum_{i=1}^6 J_i. \end{aligned} \tag{4.37}$$

The right-hand side of (4.37) will be estimated term by term as follows, using Lemma 2.1, 4.1–4.5 and Cauchy-Schwarz inequality as well.

$$\begin{aligned} J_1 &\leq \|u\|_{L^\infty} \|\rho^{1/2} u_{tt}\|_{L^2} \|\nabla u_{tt}\|_{L^2} \leq \delta \|\nabla u_{tt}\|_{L^2}^2 + C(\delta) \|\rho^{1/2} u_{tt}\|_{L^2}^2, \\ J_2 &\leq C (\|\rho u_t\|_{L^3} + \|u\|_{L^\infty} \|\rho_t\|_{L^3}) (\|\nabla u_t\|_{L^2} \|u_{tt}\|_{L^6} + \|u_t\|_{L^6} \|\nabla u_{tt}\|_{L^2}) \\ &\leq C \left(1 + \|\rho^{1/2} u_t\|_{L^2}^{1/2} \|\nabla u_t\|_{L^2}^{1/2}\right) \|\nabla u_t\|_{L^2} \|\nabla u_{tt}\|_{L^2} \\ &\leq \delta \|\nabla u_{tt}\|_{L^2}^2 + C(\delta) (1 + \|\nabla u_t\|_{L^2}^3), \\ J_3 &\leq C (\|\rho_{tt}\|_{L^2} \|u\|_{L^\infty} + \|\rho_t\|_{L^3} \|u_t\|_{L^6}) \|\nabla u\|_{L^3} \|u_{tt}\|_{L^6} \\ &\leq \delta \|\nabla u_{tt}\|_{L^2}^2 + C(\delta) (\|\rho_{tt}\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2), \\ J_4 &\leq C \|\rho^{1/2} u_{tt}\|_{L^2} \|\nabla u\|_{L^3} \|u_{tt}\|_{L^6} \leq \delta \|\nabla u_{tt}\|_{L^2}^2 + C(\delta) \|\rho^{1/2} u_{tt}\|_{L^2}^2, \\ J_5 &\leq C \|P_{tt}\|_{L^2} \|\nabla u_{tt}\|_{L^2} \leq \delta \|\nabla u_{tt}\|_{L^2}^2 + C(\delta) \|P_{tt}\|_{L^2}^2 \end{aligned}$$

and

$$\begin{aligned} J_6 &\leq C \left(\|\nabla d\|_{L^\infty} \|\nabla d_{tt}\|_{L^2} + \|\nabla d_t\|_{L^2}^{1/2} \|\nabla d_t\|_{L^6}^{3/2} \right) \|\nabla u_{tt}\|_{L^2} \\ &\leq \delta \|\nabla u_{tt}\|_{L^2}^2 + C(\delta) (\|\nabla d_{tt}\|_{L^2}^2 + \|\nabla^2 d_t\|_{L^2}^3), \end{aligned}$$

since direct computations give

$$|M(d)_{tt}| \leq C (|\nabla d| |\nabla d_{tt}| + |\nabla d_t|^2).$$

Thus, putting the estimates of J_1, \dots, J_6 into (4.37) and multiplying the resulting inequality by σ^2 , we have by choosing $\delta > 0$ small enough and using Lemmas 4.1–4.4 that

$$\begin{aligned} & \sup_{0 \leq t \leq T} \left(\sigma^2 \|\rho^{1/2} u_{tt}\|_{L^2}^2 \right) + \int_0^T \sigma^2 \|\nabla u_{tt}\|_{L^2}^2 dt \\ & \leq C + C \int_0^T \sigma \|\rho^{1/2} u_{tt}\|_{L^2}^2 dt + C \int_0^T (\|P_{tt}\|_{L^2}^2 + \|\rho_{tt}\|_{L^2}^2 + \sigma \|\nabla d_{tt}\|_{L^2}^2) dt \end{aligned}$$

$$\begin{aligned}
& + C \sup_{0 \leq t \leq T} \sigma^{1/2} (\|\nabla u_t\|_{L^2} + \|\nabla^2 d_t\|_{L^2}) \int_0^T (\|\nabla u_t\|_{L^2}^2 + \|\nabla^2 d_t\|_{L^2}^2) dt \\
& \leq C.
\end{aligned} \tag{4.38}$$

As a result of (4.38), (4.26) and Lemma 4.4, we also see that

$$\sigma \|\nabla^2 u_t\|_{L^2} \leq C \sigma \left(1 + \|\nabla u_t\|_{L^2} + \|\rho^{1/2} u_{tt}\|_{L^2} + \|\nabla^2 d_t\|_{L^2} \right) \leq C, \tag{4.39}$$

and thus, it follows from (4.32), (4.39), Lemmas 4.4 and 4.5 that for any $q \in (3, 6)$,

$$\begin{aligned}
\sigma \|\nabla^2 u\|_{W^{1,q}} & \leq C \sigma (1 + \|\nabla u_t\|_{L^2} + \|\nabla u_t\|_{L^q} + \|\nabla u\|_{H^2} + \|\nabla^2 d\|_{H^2} + \|\nabla^2 P\|_{L^q}) \\
& \leq C \sigma (1 + \|\nabla u_t\|_{H^1} + \|\nabla u\|_{H^2} + \|\nabla^2 d\|_{H^2} + \|\nabla^2 P\|_{L^q}) \\
& \leq C.
\end{aligned} \tag{4.40}$$

Therefore, combining (4.38)–(4.40) immediately leads to (4.36). \square

5 Proof of Theorem 1.1

With all the a priori estimates at hand, we are now ready to prove our main results. To this end, we first need the following local existence theorem of classical solutions of (1.1)–(1.7) with large initial data.

Proposition 5.1 *Assume that the initial data (ρ_0, u_0, d_0) satisfy the conditions (1.8), (1.9) of Theorem 1.1. Then there exist a positive time $T_0 > 0$ and a unique classical solution (ρ, u, d) of (1.1)–(1.7) on $\mathbb{R}^3 \times (0, T_0]$, satisfying $\rho \geq 0$, $|d| = 1$, and for any $\tau \in (0, T_0)$,*

$$\begin{cases}
(\rho - \tilde{\rho}, P(\rho) - P(\tilde{\rho})) \in C([0, T_0]; H^1 \cap W^{1,q}), \\
(\rho - \tilde{\rho}, P(\rho) - P(\tilde{\rho})) \in L^\infty(0, T_0; H^2 \cap W^{2,q}) \\
u \in C([0, T]; D^1 \cap D^2) \cap L^\infty(\tau, T_0; D^3 \cap D^{3,q}), \\
u_t \in L^\infty(\tau, T_0; D^1 \cap D^2) \cap H^1(\tau, T_0; D^1), \\
\nabla d \in C([0, T_0]; H^2) \cap L^\infty(\tau, T_0; H^3), \\
d_t \in C([0, T_0]; H^1) \cap L^\infty(\tau, T_0; H^2).
\end{cases} \tag{5.1}$$

Proof. As that in [20], we can use the Galerkin's approximation method to construct the approximate solutions u^m to the momentum equation, then use this approximate u^m and the equations of conservation mass and angular momentum to get ρ^m, d^m . The existence of a smooth approximate solution (ρ^m, u^m, d^m) follows from the fixed point theorem, similar to that on the compressible Navier-Stokes equations (see, for example, [20, 36]). Now, in order to prove the convergence for the approximate solutions and to obtain a smooth solution of (1.1)–(1.7), it is essential to derive some uniform a priori estimate for (ρ^m, u^m, d^m) .

It has been shown in [20, Theorem 2.1 or (3.21), (3.26)] that there exists a small $T_0 > 0$, independent of m and the lower bound of density, such that

$$\sup_{0 \leq t \leq T_0} (\|\sqrt{\rho^m} u_t^m\|_{L^2}^2 + \|\rho^m - \tilde{\rho}\|_{H^1 \cap W^{1,q}}^2 + \|\nabla u^m\|_{H^1}^2 + \|d_t^m\|_{H^1}^2 + \|\nabla d^m\|_{H^2}^2)$$

$$+ \int_0^{T_0} (\|u^m\|_{D^{2,q}}^2 + \|\nabla u_t^m\|_{L^2}^2 + \|\nabla^4 d^m\|_{L^2}^2 + \|\nabla^2 d_t^m\|_{L^2}^2) \leq \tilde{C} \exp(\tilde{C}\|g\|_{L^2}^2), \quad (5.2)$$

where $\tilde{C} > 0$ may depend on T_0 , but is independent of the size of domain.

In view of the bounds in (5.2), we can proceed to derive more higher-order estimates on the solutions (ρ^m, u^m, d^m) in the same way as those carried out in Lemmas 4.3–4.6. To summarize up, for any $\tau \in (0, T_0]$ the approximate solutions (ρ^m, u^m, d^m) satisfy

$$\begin{aligned} & \sup_{0 \leq t \leq T_0} (\|\nabla \rho^m\|_{W^{1,q}} + \|\nabla P^m\|_{W^{1,q}}) + \int_0^{T_0} (\|\nabla u_t^m\|_{L^q}^{p_0} + \|\nabla^2 u^m\|_{W^{1,q}}^{p_0}) dt \\ & + \sup_{\tau \leq t \leq T_0} (\|\sqrt{\rho^m} u_{tt}^m\|_{L^2}^2 + \|\nabla^2 u^m\|_{H^1 \cap W^{1,q}}^2 + \|\nabla u_t^m\|_{H^1}^2) + \int_\tau^{T_0} \|\nabla u_{tt}^m\|_{L^2}^2 dt \\ & + \sup_{\tau \leq t \leq T_0} (\|\nabla^2 d^m\|_{H^2}^2 + \|\nabla^2 d_t^m\|_{L^2}^2) + \int_\tau^{T_0} \|\nabla d_{tt}^m\|_{L^2}^2 dt \leq \tilde{C} \end{aligned} \quad (5.3)$$

for some positive constant \tilde{C} which may depend on T_0 , but is independent of m and the lower bound of density.

With the help of the local estimates (5.2) and (5.3), we easily deduce after taking a subsequence $(\rho^{m_j}, u^{m_j}, d^{m_j})$ and passing to limit as $j \rightarrow \infty$ that $(\rho^{m_j}, u^{m_j}, d^{m_j})$ would converges to a solution (ρ, u, d) of (1.1)–(1.7) on $\mathbb{R}^3 \times (0, T_0)$ satisfying (5.1) due to the lower semi-continuity. The uniqueness of strong/classical solutions can be proved in the same manner as that in [21]. This finishes the proof of Proposition 5.1. \square

Proof of Theorem 1.1. By Proposition 5.1, there exists a small time $T_0 > 0$ such that the Cauchy problem (1.1)–(1.7) has a unique classical solution (ρ, u, d) on $\mathbb{R}^3 \times (0, T_0]$. We shall make use of the a priori estimates, Proposition 3.1 and Lemmas 4.1–4.6, to extend the local classical solution (ρ, u, d) to all time.

First, in view of the definitions in (3.1)–(3.5), it is easily seen from (3.20) and (3.87) that

$$A_1(0) + A_2(0) = 0, \quad A_3(0) \leq C_0^{\delta_0}, \quad A_4(0) + A_5(0) \leq C_0^{\delta_0} \quad \text{and} \quad \rho_0 \leq \bar{\rho},$$

due to $C_0 \leq \varepsilon$. Thus, there exists a $T_1 \in (0, T_0]$ such that (3.7) hold for $T = T_1$.

To be continued, we set

$$T^* = \sup \{T \mid (3.7) \text{ holds}\}. \quad (5.4)$$

Then $T^* \geq T_1 > 0$. Hence, for any $0 < \tau < T \leq T^*$, it follows from (1.1), (4.4), (4.12), (4.13), (4.20), (4.36) and Lemma 2.1 that

$$\begin{aligned} & \int_\tau^T \int |\partial_t (\rho |u_t|^2)| dx dt + \int_\tau^T \int |\partial_t (\rho |u \nabla u|^2)| dx dt \\ & \leq C \int_\tau^T \int (|\rho_t| |u_t|^2 + \rho |u_t| |u_{tt}|) dx dt \\ & + C \int_\tau^T \int (|\rho_t| |u|^2 |\nabla u|^2 + \rho |u| |u_t| |\nabla u|^2 + \rho |u|^2 |\nabla u| |\nabla u_t|) dx dt \\ & \leq C \int_\tau^T \int (\rho |\operatorname{div} u| |u_t|^2 + |u| |\nabla \rho| |u_t|^2 + \rho |u_t| |u_{tt}|) dx dt \\ & + C \int_\tau^T \int (|\operatorname{div} u| |u|^2 |\nabla u|^2 + |\nabla \rho| |u|^3 |\nabla u|^2 + |u|^2 |\nabla u| |\nabla u_t| + \rho |u| |u_t| |\nabla u|^2) dx dt \end{aligned}$$

$$\begin{aligned}
&\leq C \int_{\tau}^T \left(\|\nabla u\|_{L^{\infty}} \|\rho^{1/2} u_t\|_{L^2}^2 + \|u\|_{L^6} \|\nabla \rho\|_{L^2} \|u_t\|_{L^6}^2 + \|\rho^{1/2} u_t\|_{L^2} \|\rho^{1/2} u_{tt}\|_{L^2} \right) dt \\
&\quad + C \int_{\tau}^T \left(\|\nabla u\|_{H^2}^5 + \|\nabla u\|_{H^2}^5 \|\nabla \rho\|_{L^2} + \|\nabla u\|_{H^2}^3 \|\nabla u_t\|_{L^2} + \|\rho^{1/2} u_t\|_{L^2} \|\nabla u\|_{H^2}^3 \right) dt \\
&\leq C(\tau, T),
\end{aligned}$$

which, together with (4.4) and (4.12), yields

$$\rho^{1/2} u_t, \quad \rho^{1/2} u \cdot \nabla u \in C([\tau, T]; L^2).$$

and consequently,

$$\rho^{1/2} \dot{u} \in C([\tau, T]; L^2). \quad (5.5)$$

Next, we claim that

$$T^* = \infty. \quad (5.6)$$

Otherwise, $T^* < \infty$. Then by Proposition 3.1, (3.9) holds for $T = T^*$. So, it follows from Lemmas 4.1–4.6 and (5.5) that $(\rho, u, d)(x, T^*)$ satisfies (1.11) and (1.9), where $g(x) = \rho^{1/2} \dot{u}(x, T^*)$. Thus, Proposition 5.1 implies that there exists some $T^{**} > T^*$ such that (3.7) holds for $T = T^{**}$, which contradicts (5.4). Hence, (5.6) holds. Proposition 5.1 and Lemmas 4.1–4.6 thus show that (ρ, u, d) is in fact a unique classical solution on $\mathbb{R}^3 \times (0, T]$ for any $0 < T < \infty$.

In order to complete the proof of the existence part, it remains to prove that (ρ, u, d) is continuous in t , especially, to prove that

$$(\rho - \tilde{\rho}, P(\rho) - P(\tilde{\rho})) \in C([0, T]; D^2 \cap D^{2,q}), \quad q \in (3, 6), \quad (5.7)$$

since by virtue of Lemmas 4.1–4.5 one easily deduces from (1.1)–(1.3) that

$$\begin{cases} (\rho - \tilde{\rho}, P(\rho) - P(\tilde{\rho})) \in C([0, T]; H^1 \cap W^{1,q}), \\ (\rho - \tilde{\rho}, P(\rho) - P(\tilde{\rho})) \in C([0, T]; H^2 \cap W^{2,q} - \text{weakly}), \\ (u, \nabla d) \in C([0, T]; D^1 \cap D^2). \end{cases} \quad (5.8)$$

To prove (5.7), we denote by $D_{ij} \triangleq \partial_{ij}^2$ with $i, j = 1, 2, 3$. Then it follows from (1.1) that

$$\partial_t D_{ij} \rho + \operatorname{div}(u D_{ij} \rho) = -\operatorname{div}(\rho D_{ij} u) - \operatorname{div}(\partial_i \rho \cdot \partial_j u + \partial_j \rho \cdot \partial_i u)$$

holds in $\mathcal{D}'(\mathbb{R}^3 \times (0, T))$. Now, let $j_{\nu}(x)$ is the standard mollifying kernel with width ν and set $\rho^{\nu} \triangleq \rho * j_{\nu}$. Then, we infer from the above equation that

$$\partial_t D_{ij} \rho^{\nu} + \operatorname{div}(u D_{ij} \rho^{\nu}) = -\operatorname{div}(\rho D_{ij} u) * j_{\nu} - \operatorname{div}(\partial_i \rho \cdot \partial_j u + \partial_j \rho \cdot \partial_i u) * j_{\nu} + R_{\nu}, \quad (5.9)$$

where $R_{\nu} \triangleq \operatorname{div}(u D_{ij} \rho^{\nu}) - \operatorname{div}(u D_{ij} \rho) * j_{\nu}$ satisfies (cf. [31, Lemma 2.3])

$$\int_0^T \|R_{\nu}\|_{L^2 \cap L^q}^{p_0} dt \leq C \int_0^T \|u\|_{W^{1,\infty}}^{p_0} \|D_{ij} \rho\|_{L^2 \cap L^q}^{p_0} dt \leq C \quad (5.10)$$

due to (4.29), where $p_0 > 1$ being the same one as in (4.30).

Multiplying (5.9) by $q|D_{ij} \rho^{\nu}|^{q-2} D_{ij} \rho^{\nu}$ and integrating by parts over \mathbb{R}^3 , we obtain

$$\frac{d}{dt} \|D_{ij} \rho^{\nu}\|_{L^q}^q = -(q-1) \int |D_{ij} \rho^{\nu}|^q \operatorname{div} u dx - q \int \operatorname{div}(\rho D_{ij} u) * j_{\nu} |D_{ij} \rho^{\nu}|^{q-2} D_{ij} \rho^{\nu} dx$$

$$\begin{aligned}
& -q \int (\operatorname{div}(\partial_i \rho \cdot \partial_j u + \partial_j \rho \cdot \partial_i u) * j_\nu) |D_{ij} \rho^\nu|^{q-2} D_{ij} \rho^\nu dx \\
& + q \int R_\nu |D_{ij} \rho^\nu|^{q-2} D_{ij} \rho^\nu dx,
\end{aligned}$$

which, combining with (4.29) and (5.10), yields

$$\begin{aligned}
& \sup_{0 \leq t \leq T} \|\nabla^2 \rho^\nu\|_{L^q} + \int_0^T \left| \frac{d}{dt} \|\nabla^2 \rho^\nu\|_{L^q}^q \right|^{p_0} dt \\
& \leq C + C \int_0^T (\|\nabla u\|_{W^{2,q}}^{p_0} + \|R_\nu\|_{L^2 \cap L^q}^{p_0}) dt \\
& \leq C.
\end{aligned}$$

This, together with Ascoli-Arzelà theorem, gives

$$\|\nabla^2 \rho^\nu(\cdot, t)\|_{L^q} \rightarrow \|\nabla^2 \rho(\cdot, t)\|_{L^q} \quad \text{in } C([0, T]) \quad \text{as } \nu \rightarrow 0.$$

In particular, we have

$$\|\nabla^2 \rho(\cdot, t)\|_{L^q} \in C([0, T]). \quad (5.11)$$

Similarly, one also has

$$\|\nabla^2 \rho(\cdot, t)\|_{L^2} \in C([0, T]),$$

which, combined with (5.11) and (5.8)₂, implies

$$\nabla^2 \rho \in C([0, T]; L^2 \cap L^q). \quad (5.12)$$

In the same way, we can also prove that $\nabla^2 P(\rho) \in C([0, T]; L^2 \cap L^q)$. This, together with (5.12), finishes the proof of (5.7), and thus, the existence of a classical solution (ρ, u, d) of (1.1)–(1.7) has been proved.

Next we prove the large-time behavior of (ρ, u, d) . This can be done as the ones in [16, 17], however, for completeness we sketch the proof here. Multiplying (3.35) by $4(P(\rho) - P(\tilde{\rho}))^3$ and integrating it over \mathbb{R}^3 , we get that

$$\frac{d}{dt} \|P(\rho) - P(\tilde{\rho})\|_{L^4}^4 = \int (|P(\rho) - P(\tilde{\rho})|^4 \operatorname{div} u - 3\gamma P(\rho)(P(\rho) - P(\tilde{\rho}))^3 \operatorname{div} u) dx,$$

which, together with (3.94) and (3.95), shows

$$\int_1^\infty \left| \frac{d}{dt} \|P(\rho) - P(\tilde{\rho})\|_{L^4}^4 \right| dt \leq C \int_1^\infty (\|\nabla u\|_{L^4}^4 + \|P(\rho) - P(\tilde{\rho})\|_{L^4}^4) dt \leq C.$$

As a result, we have

$$\|P(\rho) - P(\tilde{\rho})\|_{L^4} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

This, together with (3.11) and the uniform upper bound of density, shows that

$$\lim_{t \rightarrow \infty} \|\rho - \tilde{\rho}\|_{L^p} = 0 \quad (5.13)$$

holds for any p as in (1.14).

To study the large-time behavior of the velocity, we set

$$M(t) \triangleq \frac{\mu}{2} \|\nabla u\|_{L^2}^2 + \frac{\mu + \lambda}{2} \|\operatorname{div} u\|_{L^2}^2.$$

Then, multiplying (1.2) by \dot{u} in L^2 and integrating by parts over \mathbb{R}^3 , we obtain

$$- \int (\mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u) \cdot \dot{u} dx = \int ((P(\rho) - P(\tilde{\rho})) \operatorname{div} \dot{u} + M(d) : \nabla \dot{u} - \rho |\dot{u}|^2) dx. \quad (5.14)$$

Recalling the definition of “ \cdot ”, we deduce after integrating by parts that

$$\begin{aligned} - \int \mu \Delta u \cdot \dot{u} dx &= \frac{\mu}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \mu \int \partial_k u^j \partial_k (u_i \partial_i u^j) dx \\ &= \frac{\mu}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \mu \int \left(\partial_k u^j \partial_k u_i \partial_i u^j - \frac{1}{2} |\nabla u|^2 (\operatorname{div} u) \right) dx, \end{aligned}$$

and similarly,

$$- \int (\mu + \lambda) \nabla \operatorname{div} u \cdot \dot{u} dx = \frac{\mu + \lambda}{2} \frac{d}{dt} \|\operatorname{div} u\|_{L^2}^2 + O(\|\nabla u\|_{L^3}^3),$$

which, inserted into (5.14), yields

$$\begin{aligned} |M'(t)| &\leq C \left(\|\nabla u\|_{L^3}^3 + \|\rho^{1/2} \dot{u}\|_{L^2}^2 + (\|P(\rho) - P(\tilde{\rho})\|_{L^2} + \|\nabla d\|_{L^3} \|\nabla d\|_{L^6}) \|\nabla \dot{u}\|_{L^2} \right) \\ &\leq C \left(\|\nabla u\|_{L^2} \|\nabla u\|_{L^4}^2 + \|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla \dot{u}\|_{L^2} + \|\nabla^2 d\|_{L^2}^2 \|\nabla \dot{u}\|_{L^2} \right). \end{aligned}$$

Thus, by (3.88) and (3.95) we see that

$$\begin{aligned} \int_1^\infty |M'(t)|^2 dt &\leq C \int_1^\infty \left(\|\rho^{1/2} \dot{u}\|_{L^2}^4 + \|\nabla u\|_{L^4}^4 + \|\nabla \dot{u}\|_{L^2}^2 \right) dt \\ &\leq C \left(\sup_{t \geq 1} \|\rho^{1/2} \dot{u}\|_{L^2}^2 \right) \int_1^\infty \|\rho^{1/2} \dot{u}\|_{L^2}^2 dt \leq C, \end{aligned}$$

from which and (3.88) it follows that

$$\|\nabla u(t)\|_{L^2} \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (5.15)$$

As a result, we also have

$$\int \rho^{1/2} |u|^4 dx \leq \left(\int \rho |u|^2 dx \right)^{1/2} \|u\|_{L^6} \leq C \|\nabla u\|_{L^2} \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (5.16)$$

Moreover, using (3.11), (3.15) and (3.88), we infer from (2.6) that

$$\sup_{t \geq 1} \|\nabla u(t)\|_{L^6} \leq C,$$

which, together with (5.15) and the interpolation inequality, leads to

$$\|\nabla u(t)\|_{L^r} \rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad \forall r \in [2, 6). \quad (5.17)$$

Finally, applying ∇ to both sides of (1.3) and taking the L^2 -inner product, by (3.11) and (3.88) we deduce after integrating by parts over $\mathbb{R}^3 \times (1, \infty)$ that

$$\begin{aligned} \int_1^\infty \left| \frac{d}{dt} \|\nabla^2 d\|_{L^2}^2 \right| dt &\leq C \int_1^\infty (\|\nabla d_t\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 + \|\nabla d\|_{L^6}^6 + \|\nabla d\|_{L^3}^2 \|\nabla^2 d\|_{L^6}^2) dt \\ &\quad + C \int_1^\infty (\|\nabla u\|_{L^2}^2 \|\nabla d\|_{L^\infty}^2 + \|u\|_{L^6}^2 \|\nabla^2 d\|_{L^2} \|\nabla^2 d\|_{L^6}) \\ &\leq C \int_1^\infty (\|\nabla d_t\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2) dt \\ &\leq C, \end{aligned}$$

which, together with (3.11), gives

$$\|\nabla^2 d(t)\|_{L^2} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

This, combining with (3.11) and (3.88), yields that for any $k \in (2, 6)$,

$$\|\nabla d(t)\|_{W^{1,k}} \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (5.18)$$

Combining (5.13) with (5.15)–(5.18) immediately proves (1.13). The proof of Theorem 1.1 is therefore completed. \square

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